方程的另用——从生锈圆规作图问题到生锈圆规作图方程，从 3n+1 猜想到 3n+1 方程

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论文题目：<u>**方程的另用——从生锈圆规作图问题到生锈圆规作图方程，从 3n+1 猜想到 3n+1 方程**</u>

摘要：关于生锈圆规作图（用固定半径为 1 的圆规作图）问题由来已久。10 世纪时，有数学家提出用直尺和半径固定的圆规作图。上世纪著名的美国的几何学家**佩多（Pedoe）**曾精心选择了两个问题在加拿大的一个数学杂志《Crux》上征解。两个问题均为中国数学工作者解决（见文献[1])。目前这方面最好的结果是证明了从两点出发作图时生锈圆规的能力和普通规尺是等价的（见文献[2])。中科院张景中院士曾在 1986 年中国的《自然杂志》9 卷 4 期上提出了一个未解决的生锈圆规作图问题：已知 A,B,C 三点，能否作出点 D, 使得 AC=AD,BC=BD？

本文在提出的一级，二级，三级作图的方法（也可见文献[3]、文献[4]）的基础上构造了生锈圆规作图方程（也可见文献[4]）。生锈圆规作图方程的作用是：说明哪些点为什么可以利用生锈圆规作出，哪些点为什么不可以利用生锈圆规作出。本文提出了一个生锈圆规作图方程定理，总结了前人为什么可以证明从两点出发作图时生锈圆规的能力和普通规尺是等价的，以及一些尚未解决的问题的解答，即问题（1）：已知 A,B,C 三点，能否作出点 D, 使得 AC=AD,BC=BD？（张景中院士曾在 1986 年中国的《自然杂志》9 卷 4 期上提出）问题（2）：已知 A，B，C，D 四点（四个点中任意三点不共线），能否作出直线 AB 与 CD 的交点 E？直线 AC 与 BD 的交点 F？直线 AD 与 BC 交点 G？问题（3）：仅利用$\triangle ABC$的三个顶点 A,B,C，能否作出$\triangle ABC$的

外心点 D、内心点 E、垂心点 F、三个垂足点G_1,点G_2,点G_3、三个旁心点H_1,点H_2,点H_3？结论认为这三个问题在一般情况下都是不能做出的，并在此基础上提出了一个作图定理:若平面上存在 n+1 个点,分别设为：$(a,b),(a+x_1,b+y_1),\cdots,(a+x_n,b+y_n)$，且不对$(x_1,y_1),\cdots,(x_n,y_n)$进行特殊限定，则对于坐标为：

$$(a+\sum_{j}^{m}(A_j\bullet(\sum_{i=1}^{n}{}_ja_i\bullet x_i+{}_jb_i\bullet y_i),\quad b+\sum_{j}^{m}(B_j\bullet(\sum_{i=1}^{n}{}_jc_i\bullet x_i+{}_jd_i\bullet y_i))$$

(${}_ja_i,{}_jb_i,{}_jc_i,{}_jd_i\in R$，$A_j$，$B_j$均是关于${}_je_i\bullet x_i$，${}_jf_i\bullet y_i$进行有限次运算，乘方，开方运算得到的代数式，且不能化为一个常数。(${}_je_i$，${}_jf_i\in R$))这样的点利用生锈圆规作图均不能作出，利用尺规作图或圆规作图部分点可以做出。

考拉兹猜想，又称为 3n＋1 猜想、角谷猜想、哈塞猜想、乌拉姆猜想或叙拉古猜想，是指对於每一个正整数，如果它是奇数，则对它乘 3 再加 1，如果它是偶数，则对它除以 2，如此循环，最终都能够得到 1。作者在利用生锈圆规作图方程解决了生锈圆规作图问题后，猜测 3n+1 猜想也可以通过构造方程的方法解决，因此构造了 3n+1 方程，并通过 3n+1 方程解决 3n+1 猜想但还没有成功，但作者仍在努力中。

Paper Title:Another use of Equations——From the Problem of Construction with Rusty Compasses to the equation of Construction with Rusty Compasses，From the Problem of Construction with 3n+1 to the equation of Construction with 3n+1

Abstract: The problem of construction with rusty compasses (its fixed radius is 1) has been so for quite some time. The 10th century ,mathematicians have proposed using a ruler and compass fixed radius plot.A famous American geometrician---Pedoe once selected two problems to collect solutions in a Canadian mathematical magazine "Crux" last century. Both of the two problems were solved by Chinese mathematicians (See reference [1]). At present, the best result in this respect is to verify that the ability of rusty compasses is equivalent to that of common ruler and compasses when constructing from two points (See reference [2]). Chinese academy of sciences Zhang Jing Zhong academician worked in the China's 《Nature magazine》9 volume 4 make a not solved Problem: Make use of Rusty Compasses ,If three points A, B and C are known, can you construct a point D to make AC=AD and BC=BD?

This paper on one-level, two-level and three-level basis , structure goes out the equation of Construction with Rusty Compasses. The equation of Construction with Rusty Compasses effect is the explaining that why which point can make use of rusty compasses is made, why which point can not make use of rusty compasses make. This paper have summed up prehominid why to be able to certificate to verify that the ability of rusty compasses is equivalent to that of common ruler and compasses when constructing from two points ,and answering relate to some problems hanging but not having untied ，namely problem （1）: If three points A, B and C are known, can you construct a point D to make AC=AD and BC=BD? Problem （2）: If four points A, B, C and D are known (Three random points in the four points are not collinear), can you construct the intersection point E of straight line AB and CD? Can you construct the intersection point F of straight line AC and BD? And can you construct the intersection point G of straight line AD and BC? Problem （3）: Make use of only of the $\triangle ABC$ three point A,B,C，can you construct of the $\triangle ABC$ circumcenter point D、heart point E、orthocenter point

F、three foot point point G_1, point G_2, point G_3、three other heart point H_1, point H_2, point H_3。The conclusion showed that neither of the some problems could be solved（on the general situation）.And on this basis ,proposed a mapping theorem: If the plane of existence of n+1 points, were set to:

$(a,b),(a+x_1,b+y_1),\cdots,(a+x_n,b+y_n)$,and not on ,

$(x_1,y_1),\cdots,(x_n,y_n)$ conduct special limit ,then the coordinates is ：

$$(a+\sum_{j}^{m}(A_j\bullet(\sum_{i=1}^{n}{}_ja_i\bullet x_i+{}_jb_i\bullet y_i),\quad b+\sum_{j}^{m}(B_j\bullet(\sum_{i=1}^{n}{}_jc_i\bullet x_i+{}_jd_i\bullet y_i))$$

$({}_ja_i,{}_jb_i,{}_jc_i,{}_jd_i\in R,\ \ A_j,B_j$ are on ${}_je_i\bullet x_i,{}_jf_i\bullet y_i$ limited time computing ,power , square root to be algebraic ,and can not be turned into a constant .$({}_je_i,{}_jf_i\in R))$ this point using rusty compasses can not be made ,using ruler and compass or compass mapping part of the plot points can be made .

3n+1 conjecture ， tells us that for arbitrary positive integer, if it's an odd number, we make a manipulation——multiplies 3 and then plus 1;if it's an even number, we make a manipulation——divided by 2. Circulating in this way, we may get 1 at last. After solving the problem ofconstruction with rusty compasses, the author thought that it would be similar to solve 3n+1 conjecture by constructing 3n+1 equation. The author has successfully constructed 3n+1 equation and tries his best to solve it finally, however, it doesn't be solved now. But authors are still work in the middle.

目录

一、生锈圆规（固定半径为 1 的圆规）的基本作图、基本方法：

基本作图：

利用平面上一个点（a,b）可以得到的一个以点（a,b）为圆心的单位圆。

基本方法：

1、当两固定点的距离等于 2 时，可得一个新的固定点，即原来两点的中点。

2、当两固定点距离小于 2 时，可作出两个新的固定点，与原来两点构成边长为 1 的菱形。

二、利用的公式、规定和一级，二级，三级作图以及其目的，特点：

公式：设点 $A(x_1, y_1)$，点 $B(x_2, y_2)$，且线段长度不大于 2，用生锈圆规分别以点 $A(x_1, y_1)$，点 $B(x_2, y_2)$ 为圆心作单位圆可直接作出点

$$\left(\frac{x_1+x_2}{2}+\sqrt{\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}-\frac{1}{4}}\bullet(y_1-y_2),\frac{y_1+y_2}{2}+\sqrt{\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}-\frac{1}{4}}\bullet(x_2-x_1)\right)$$

和点

$$\left(\frac{x_1+x_2}{2}-\sqrt{\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}-\frac{1}{4}}\bullet(y_1-y_2),\frac{y_1+y_2}{2}-\sqrt{\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}-\frac{1}{4}}\bullet(x_2-x_1)\right)$$

两点可重合。

则所得的点分别与点 A，点 B 的坐标差表达式为：

$$\left(\pm\left(\frac{x_2-x_1}{2}+\sqrt{\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}-\frac{1}{4}}\bullet(y_1-y_2)\right),\pm\left(\frac{y_2-y_1}{2}+\sqrt{\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}-\frac{1}{4}}\bullet(x_2-x_1)\right)\right)$$

和

$$\left(\pm\left(\frac{x_1-x_2}{2}+\sqrt{\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}-\frac{1}{4}}\bullet(y_1-y_2)\right),\pm\left(\frac{y_1-y_2}{2}+\sqrt{\frac{1}{(x_1-x_2)^2+(y_1-y_2)^2}-\frac{1}{4}}\bullet(x_2-x_1)\right)\right)$$

可相同。

规定：1、对本文的的所有式子里所有出现的“$\cos x$”和“$\sin x$” 而言，$x\in(0,2\pi)$。

2、因为点 $(a+\cos x, b+\sin x)$ 在以点 (a,b) 为圆心的单位圆中有唯一的一个点与之对应，所以用关于“$\cos x$”和“$\sin x$”的代数式表示点的坐标表达式。因此规定两点距离为 1 的两点的横坐标差表达式为“A1”， 纵横坐标差表达式为“B1”， 两点距离为 1 的两点的坐标差表达式为（A1， B1）。（（A1， B1）代表的是 $(\cos x, \sin x)$。）

3、若平面上有两单位圆有交点，则设两圆中一圆的圆心坐标为 (x, y)，两单位圆交点为点

$(x+\cos\theta_1, y+\sin\theta_1)$，点 $(x+\cos\theta_2, y+\sin\theta_2)$，则另一圆的圆心坐标为 $(x+\cos\theta_1+\cos\theta_2, y+\sin\theta_1+\sin\theta_2)$。（两圆相切时 $\theta_1=\theta_2$，即两点重合。）

4、利用生锈圆规，从点 $A(x, y)$ 和以点 A 为圆心的单位圆上一点 $(x+\cos\theta, y+\sin\theta)$ 出发就可得：点列 $(x+\cos(\theta+\frac{n\pi}{3}), y+\sin(\theta+\frac{n\pi}{3})), (n$为0，1，2，3，4，5)，点列中的六个点是以点 A 为圆心的单位圆的一个内接正六边形。因为点列 $(x+\cos(\theta+\frac{n\pi}{3}), y+\sin(\theta+\frac{n\pi}{3})), (n$为0，1，2，3，4，5)中的这六个点分别与点 $A(x, y)$ 的坐标差是六个（A1，B1），即坐标差列 $(\cos(\theta+\frac{n\pi}{3}), \sin(\theta+\frac{n\pi}{3})), (n$为0，1，2，3，4，5)中的六个元素，于是规定坐标差列 $(\cos(\theta+\frac{n\pi}{3}), \sin(\theta+\frac{n\pi}{3})), (n$为0，1，2，3，4，5)为（6A1，6B1）。

5、当平面上有两个点时：利用 $(x, y), (x+\cos\theta_1, y+\sin\theta_1)$，$(x+\cos\theta_2, y+\sin\theta_2)$，$(x+\cos\theta_1+\cos\theta_2, y+\sin\theta_1+\sin\theta_2)$ 这样的形式。设其中的一点坐标表达式为 (x, y)，在以 (x, y) 点为圆心的单位圆取两点设为点 $(x+\cos a_1, y+\sin a_1)$ 和点 $(x+\cos a_2, y+\sin a_2)$，此三点可确定点 $(x+\cos a_1+\cos a_2, y+\sin a_1+\sin a_2)$。在以点 $(x+\cos a_1+\cos a_2, y+\sin a_1+\sin a_2)$ 为圆心的单位圆取两点设为点 $(x+\cos a_1+\cos a_2+\cos a_3, y+\sin a_1+\sin a_2+\sin a_3)$ 和点 $(x+\cos a_1+\cos a_2+\cos a_4, y+\sin a_1+\sin a_2+\sin a_4)$，此三点可确定点 $(x+\cos a_1+\cos a_2+\cos a_3+\cos a_4, y+\sin a_1+\sin a_2+\sin a_3+\sin a_4)$，在以点 $(x+\cos a_1+\cos a_2, y+\sin a_1+\sin a_2)$ ·······。利用这个方法连接平面上的另一个点，即利用这个方法直到与以其它点为圆心的单位圆有交点即可。因此可设平面上的另一点的坐标表达式均为 $(x+\sum_{i=1}^{2n}\cos a_i, y+\sum_{i=1}^{2n}\sin a_i)$ 这样的形式。（注：作图时连接所利用的辅助点随时可变，所以表达式不唯一，因此用这个通式表示。）

6、因为在作图过程中得到了这些辅助点：点$(x+\cos a_1, y+\sin a_1)$，点$(x+\cos a_2, y+\sin a_2)$，点$(x+\cos a_1+\cos a_2, y+\sin a_1+\sin a_2)$，点$(x+\cos a_1+\cos a_2+\cos a_3, y+\sin a_1+\sin a_2+\sin a_3)$，••••••，点$(x+\sum_{i=1}^{2n-1}\cos a_i, y+\sum_{i=1}^{2n-1}\sin a_i)$,点$(x+\sum_{i=1}^{2n}\cos a_i, y+\sum_{i=1}^{2n}\sin a_i)$，这些辅助点是在作图过程中连接而得，所以将辅助点分为两种类型：非作图时连接而得型辅助点（简称为非连接型辅助点)，作图时连接而得型辅助点(简称为连接型辅助点)。

非连接型辅助点与连接型辅助点的区别：连接型辅助点是随时可变的，连接型辅助点确定“原始”的（A1， B1）。 非连接型辅助点在作图中可有可无而且不能完全满足这些性质。

非连接型辅助点的作用：已知平面上存在 a 个点，再利用平面上 b 个非连接型辅助点，然后利用连接型辅助点连接后，讨论从这 a+b 个点出发作图和从这 a 个点作图的联系。

连接时所得的的所有（A1， B1）是否会出现属于同一（6A1，6 B1）的情况要看实际的作图，实际的作图可以出现这种情况也可以不出现这种情况。

利用连接型辅助点进行全面连接后，根据平面上这些点中点与点之间的某些关系，可将生锈圆规作图分为一级，二级，三级作图：

第一：在平面上这些点中，当两点满足点(a, b)和点$(a+\cos\theta, b+\sin\theta)$这样的关系时，用生锈圆规分别以点$(a, b)$和点$(a+\cos\theta, b+\sin\theta)$为圆心作单位圆确定新的点，得到点集：

$(a+\frac{x}{2}\cos\theta-\frac{\sqrt{3}y}{2}\sin\theta, b+\frac{x}{2}\sin\theta+\frac{\sqrt{3}y}{2}\cos\theta)$（$x, y\in z, x, y$ 同奇偶）这样的形式。

由$(a+\sum_{i=1}^{6}x_i\cos(\theta+\frac{(i-1)\pi}{3}), b+\sum_{i=1}^{6}x_i\sin(\theta+\frac{(i-1)\pi}{3}))$（对任意的$x_i$而言，$x_i\in z$）化简所得。

设利用点与点之间的这种关系作图为一级作图。目的：将平面上存在的每一个（A1,B1）所

属于的（6A1,6B1）中的所有元素展现在平面之上。其特点：所得（A1,B1）同属于一个（6A1,6B1）。在这个所得点的点集中，用生锈圆规分别以其中的距离不大于 2 的任意两个点为圆心作单位圆后得到的点，仍在这个点集中。这个点集的几何图形就是前人反复使用的边长为 1 的“蛛网”图。一级作图所得到的点的表达式是一个点的坐标加上多个（A1,B1），且这些（A1,B1）是同一个（6A1,6B1）中的元素。

第二：经过一级作图后，在平面上的点中，当三点 A,B,C 满足是一个边长为 1 的菱形上的三个顶点,且 AB=AC=1，点 B，点 C 不重合时，即满足点 A 坐标为(a, b)，点 B 坐标为$(a + \cos\theta_1, b + \sin\theta_1)$，点 C 坐标为$(a + \cos\theta_2, b + \sin\theta_2)$这样的形式（$\theta_1$，$\theta_2 \in (0, 2\pi)$，且$\theta_1 \neq \theta_2$）。用生锈圆规以点 B,点 C 为圆心作单位圆后得到另外一点 D,则 AB=AC=BD=CD=1。利用一级作图后，点与点之间的这种关系确定新的点。设利用点与点之间的这种关系作图为二级作图。（在此过程中用到并实验验证了恒等式：

$$((\cos a_1-\cos a_2)^2+(\sin a_1-\sin a_2)^2)(\cos a_1+\cos a_2)^2=$$
$$((\cos a_1+\cos a_2)^2+(\sin a_1+\sin a_2)^2)(\sin a_1-\sin a_2)^2$$
和
$$((\cos a_1-\cos a_2)^2+(\sin a_1-\sin a_2)^2)(\sin a_1+\sin a_2)^2=$$
$$((\cos a_1+\cos a_2)^2+(\sin a_1+\sin a_2)^2)(\cos a_1-\cos a_2)^2$$
。）

经过二级作图后，平面上所有的点的集合为：

$$(a+\sum_{i=1}^{n}(\frac{x_i}{2}\cos\theta_i-\frac{\sqrt{3}y_i}{2}\sin\theta_i),$$
$$b+\sum_{i=1}^{n}(\frac{x_i}{2}\sin\theta_i+\frac{\sqrt{3}y_i}{2}\cos\theta_i))$$

（对于任意的 x_i, y_i 而言，$x_i, y_i \in z, x_i, y_i$ 同奇偶）这样的形式。

由

$$(a+\sum_{i=1}^{6n} x_i \cos(\theta_{\left[\frac{i-1}{6}\right]+1}+\frac{(i-6\times\left[\frac{i-1}{6}\right]-1)\pi}{3}),$$

$$b+\sum_{i=1}^{6n} x_i \sin(\theta_{\left[\frac{i-1}{6}\right]+1}+\frac{(i-6\times\left[\frac{i-1}{6}\right]-1)\pi}{3}))$$

（对于任意的x_i,y_i而言，$x_i,y_i\in z,x_i,y_i$同奇偶，$\left[\frac{i-1}{6}\right]$为高斯函数）化简所得。

所得的（A1，B1）为是原来存在的两个不同的（6A1，6B1）的元素。目的：全面三级作图的基础。其特点：二级作图所得到的点的表达式是一级作图所得到的点的表达式加上多个（A1,B1），且这些（A1,B1）是可以不是一个（6A1,6B1）中的元素

第三：经过二级作图后，在平面上的点中，两点 A,B 距离小于 2，而且既不具备一级作图的条件，也不满足在平面上的点中，再找出一点，使得这三点具备二级作图的条件时：设点 A 的坐标为(a,b)，点 B 的坐标（已化简）为：

$$(a+\sum_{i=1}^{n}(\frac{x_i}{2}\cos\theta_i-\frac{\sqrt{3}y_i}{2}\sin\theta_i),b+\sum_{i=1}^{n}(\frac{x_i}{2}\sin\theta_i+\frac{\sqrt{3}y_i}{2}\cos\theta_i))$$

（对于任意的x_i,y_i而言，$x_i,y_i\in z,x_i,y_i$同奇偶）

用生锈圆规以点 A，点 B 为圆心作单位圆得点

$$(a+\sum_{i=1}^{n}(\frac{x_i}{4}\cos\theta_i-\frac{\sqrt{3}y_i}{4}\sin\theta_i)-\sqrt{\frac{1}{\sum_{i=1}^{n}\frac{x_i^2+3y_i^2}{4}+\sum_{1\le i<j\le n}\left(\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j)\right)}-\frac{1}{4}}\bullet(\sum_{i=1}^{n}(\frac{x_i}{2}\sin\theta_i+\frac{\sqrt{3}y_i}{2}\cos\theta_i)),$$

$$b+\sum_{i=1}^{n}\left(\frac{x_i}{4}\sin\theta_i+\frac{\sqrt{3}y_i}{4}\cos\theta_i\right)+$$

$$\sqrt{\begin{aligned}&\frac{1}{\sum_{i=1}^{n}\frac{x_i^{\,2}+3y_i^{\,2}}{4}}-\frac{1}{4}\bullet(\sum_{i=1}^{n}(\frac{x_i}{2}\cos\theta_i-\frac{\sqrt{3}y_i}{2}\sin\theta_i)))\\&+\sum_{1\le i<j\le n}\begin{matrix}(\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)+\\ \frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j))\end{matrix}\end{aligned}}$$

和点

$$(a+\sum_{i=1}^{n}\left(\frac{x_i}{4}\cos\theta_i-\frac{\sqrt{3}y_i}{4}\sin\theta_i\right)+$$

$$\sqrt{\begin{aligned}&\frac{1}{\sum_{i=1}^{n}\frac{x_i^{\,2}+3y_i^{\,2}}{4}}-\frac{1}{4}\bullet(\sum_{i=1}^{n}(\frac{x_i}{2}\sin\theta_i+\frac{\sqrt{3}y_i}{2}\cos\theta_i)),\\&+\sum_{1\le i<j\le n}\begin{matrix}(\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)+\\ \frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j))\end{matrix}\end{aligned}}$$

$$b+\sum_{i=1}^{n}\left(\frac{x_i}{4}\sin\theta_i+\frac{\sqrt{3}y_i}{4}\cos\theta_i\right)-$$

$$\sqrt{\begin{aligned}&\frac{1}{\sum_{i=1}^{n}\frac{x_i^{\,2}+3y_i^{\,2}}{4}}-\frac{1}{4}\bullet(\sum_{i=1}^{n}(\frac{x_i}{2}\cos\theta_i-\frac{\sqrt{3}y_i}{2}\sin\theta_i)))\\&+\sum_{1\le i<j\le n}\begin{matrix}(\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)+\\ \frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j))\end{matrix}\end{aligned}}$$

。所得的（A1,B1）

$$(\pm(\sum_{i=1}^{n}(\frac{x_i}{4}\cos\theta_i-\frac{\sqrt{3}y_i}{4}\sin\theta_i)-\sqrt{\frac{1}{\sum_{i=1}^{n}\frac{x_i^2+3y_i^2}{4}+\sum_{1\le i<j\le n}\left(\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j)\right)}-\frac{1}{4}}\bullet(\sum_{i=1}^{n}(\frac{x_i}{2}\sin\theta_i+\frac{\sqrt{3}y_i}{2}\cos\theta_i))),$$

为

$$\pm(\sum_{i=1}^{n}(\frac{x_i}{4}\sin\theta_i+\frac{\sqrt{3}y_i}{4}\cos\theta_i)+\sqrt{\frac{1}{\sum_{i=1}^{n}\frac{x_i^2+3y_i^2}{4}+\sum_{1\le i<j\le n}\left(\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j)\right)}-\frac{1}{4}}\bullet(\sum_{i=1}^{n}(\frac{x_i}{2}\cos\theta_i-\frac{\sqrt{3}y_i}{2}\sin\theta_i))))$$

和

$$(\pm(\sum_{i=1}^{n}(\frac{x_i}{4}\cos\theta_i-\frac{\sqrt{3}y_i}{4}\sin\theta_i)+\sqrt{\frac{1}{\sum_{i=1}^{n}\frac{x_i^2+3y_i^2}{4}+\sum_{1\le i<j\le n}\left(\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j)\right)}-\frac{1}{4}}\bullet(\sum_{i=1}^{n}(\frac{x_i}{2}\sin\theta_i+\frac{\sqrt{3}y_i}{2}\cos\theta_i))),$$

$$\pm\left(\sum_{i=1}^{n}\left(\frac{x_i}{4}\sin\theta_i+\frac{\sqrt{3}y_i}{4}\cos\theta_i\right)-\sqrt{\frac{1}{\sum_{i=1}^{n}\frac{x_i^2+3y_i^2}{4}+\sum_{1\le i<j\le n}\left(\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j)\right)}-\frac{1}{4}\bullet\left(\sum_{i=1}^{n}\left(\frac{x_i}{2}\cos\theta_i-\frac{\sqrt{3}y_i}{2}\sin\theta_i\right)\right)}\right)$$

，这些（A1,B1）不是原来存在的（A1,B1),而是新的（A1,B1)，且得到一个新的（A1,B1)就得到这一个新的（A1,B1)的（6A1,6B1）（通过 x_i，y_i 的取值和表达式可知）。因此可省去下一次的一级作图。

设利用点与点之间的这种关系的作图称为三级作图。目的：由原来存在的（A1,B1)得到新的（A1,B1)。其特点：三级作图所得到的点的表达式是一个二级作图所得到的点的表达式加上一个新的（A1,B1)。

总结:因为作图时所得到的点是在两个单位圆相交或相切时的情况下所得，所以作图所得点均可化为点

$$\begin{pmatrix} a+\sum_{i=1}^{6n}x_i\cos\left(\theta_{\left[\frac{i-1}{6}\right]+1}+\dfrac{\left(i-6\times\left[\frac{i-1}{6}\right]-1\right)\pi}{3}\right), \\ b+\sum_{i=1}^{6n}x_i\sin\left(\theta_{\left[\frac{i-1}{6}\right]+1}+\dfrac{\left(i-6\times\left[\frac{i-1}{6}\right]-1\right)\pi}{3}\right) \end{pmatrix}$$

（对于任意的 x_i,y_i 而言，$x_i,y_i\in z,x_i,y_i$ 同奇偶，$\left[\frac{i-1}{6}\right]$ 为高斯函数）。这样的形式。所以（A1,B1)是决定生锈圆规作图“关键”。

因此，将经过第一次一级，二级，三级作图后的平面上的所有点，进行第二次二级，三级

作图。将经过第二次二级，三级作图后的平面上的所有点，进行第三次二级，三级作图，……，反复下去就是生锈圆规作图系统化。

当解决在一些已知点的条件下所求的点能否作出的问题，就是利用这些已知点进行若干次一级，二级，三级作图能否得到所求的点的坐标，或利用这些已知点并另外利用一些辅助点进行若干次一级，二级，三级作图能否得到所求的点的坐标的问题。对于一些问题需要作图去实践，得到的点根据其得到的途径，便可得到其实际作图的方法。

三、利用方法一级，二级，三级作图构造生锈圆规作图方程：

由以上内容可知，由于一级作图，二级作图，三级作图各自的特点，可知利用生锈圆规作出的点，一定在二级作图所得到的点的表达式中，不一定在一级作图所得到的点的表达式和三级作图所得到的点的表达式中，所以生锈圆规作图问题可以用二级作图所得到的点的表达式解决。通过理论可推得二级作图所得到的点的表达式是一个可以化为一个关于整数与三角函数进行加，减，乘，除和开平方运算搅在一起的一个复杂的解析几何方程，称这个解析几何方程为生锈圆规作图方程。设继续到第 n+1 次一级，二级，三级作图的二级作图。另外，设第 i+1 次一级，二级，三级作图的二级作图所得的点的集合表达式为 F_i 。

设平面上存在 n 个点，其中一个点坐标为（0，0），利用连接型辅助点进行全面连接后，平面上存在 n 个（A1，B1），依次为 $(\cos\theta_1,\sin\theta_1)$，•••，$(\cos\theta_{n_0},\sin\theta_{n_0})$ 经过第一次一级、二级、三级作图的二级作图后，平面上的点的集合为（即 F_0 ）：

$$\left(\sum_{i=1}^{n_0}\left(\frac{x_i}{2}\cos\theta_i-\frac{\sqrt{3}y_i}{2}\sin\theta_i\right),\right.$$
$$\left.\sum_{i=1}^{n_0}\left(\frac{x_i}{2}\sin\theta_i+\frac{\sqrt{3}y_i}{2}\cos\theta_i\right)\right)$$

（对于任意的x_i, y_i而言，$x_i, y_i \in z, x_i, y_i$同奇偶）

=>满足第一次一级、二级、三级作图的三级作图的两点坐标差为：

$$(\sum_{i=1}^{n_0}(\frac{x_i}{2}\cos\theta_i - \frac{\sqrt{3}y_i}{2}\sin\theta_i),$$
$$\sum_{i=1}^{n_0}(\frac{x_i}{2}\sin\theta_i + \frac{\sqrt{3}y_i}{2}\cos\theta_i))$$

（对于任意的x_i, y_i而言，$x_i, y_i \in z, x_i, y_i$同奇偶）

设横坐标差为${}_1A$，纵坐标差为${}_1B$。则第一次一级、二级、三级作图的三级作图所确定（A1，

$$(\frac{{}_1A}{2}+\sqrt{\frac{1}{{}_1A^2+{}_1B^2}-\frac{1}{4}}\bullet{}_1B, \frac{{}_1B}{2}-\sqrt{\frac{1}{{}_1A^2+{}_1B^2}-\frac{1}{4}}\bullet{}_1A)\text{和}$$
$$(\frac{{}_1A}{2}-\sqrt{\frac{1}{{}_1A^2+{}_1B^2}-\frac{1}{4}}\bullet{}_1B, \frac{{}_1B}{2}+\sqrt{\frac{1}{{}_1A^2+{}_1B^2}-\frac{1}{4}}\bullet{}_1A)$$

B1）通式为：$${}_1A^2+{}_1B^2=\sum_{i=1}^{n_0}\frac{{x_i}^2+3{y_i}^2}{4}+\sum_{1\le i<j\le n_0}\frac{(x_ix_j+3y_iy_j)}{2}\cos(\theta_i-\theta_j)$$
$$+\sum_{1\le i<j\le n_0}\frac{\sqrt{3}(x_iy_j-x_jy_i)}{2}\sin(\theta_i-\theta_j)$$

将第一次一级，二级，三级作图后，平面上所存在（A1，B1）的均化为

$$(\frac{{}_1A}{2}+\sqrt{\frac{1}{{}_1A^2+{}_1B^2}-\frac{1}{4}}\bullet{}_1B, \frac{{}_1B}{2}-\sqrt{\frac{1}{{}_1A^2+{}_1B^2}-\frac{1}{4}}\bullet{}_1A)\text{和}$$
$$(\frac{{}_1A}{2}-\sqrt{\frac{1}{{}_1A^2+{}_1B^2}-\frac{1}{4}}\bullet{}_1B, \frac{{}_1B}{2}+\sqrt{\frac{1}{{}_1A^2+{}_1B^2}-\frac{1}{4}}\bullet{}_1A)$$

（${}_1A$ = $\sum_{i=1}^{n_0}(\frac{x_i}{2}\cos\theta_i - \frac{\sqrt{3}y_i}{2}\sin\theta_i)$，${}_1B$= $\sum_{i=1}^{n_0}(\frac{x_i}{2}\sin\theta_i + \frac{\sqrt{3}y_i}{2}\cos\theta_i)$（对于任意的$x_i, y_i$而言，$x_i, y_i \in z, x_i, y_i$同奇偶））这样的形式。（第一次一级，二级，三级作图的一级作图后，平面上所存在（A1，B1）在此的变换是${}_1A$= $2\cos(\theta_x+\frac{n\pi}{3})$，${}_1B$= $2\sin(\theta_x+\frac{n\pi}{3})$，

$n \in \{0,1,2,3,4,5\}$。）

设

$${}_1^1A=\sum_{i=1}^{n_0}\left(\frac{{}_1x_i^1}{2}\cos\theta_i-\frac{\sqrt{3}\,{}_1y_i^1}{2}\sin\theta_i\right),\ {}_1^1B=\sum_{i=1}^{n_0}\left(\frac{{}_1x_i^1}{2}\sin\theta_i+\frac{\sqrt{3}\,{}_1y_i^1}{2}\cos\theta_i\right).$$

………

$${}_1^{n_1}A=\sum_{i=1}^{n_0}\left(\frac{{}_1x_i^{n_1}}{2}\cos\theta_i-\frac{\sqrt{3}\,{}_1y_i^{n_1}}{2}\sin\theta_i\right),\ {}_1^{n_1}B=\sum_{i=1}^{n_0}\left(\frac{{}_1x_i^{n_1}}{2}\sin\theta_i+\frac{\sqrt{3}\,{}_1y_i^{n_1}}{2}\cos\theta_i\right).$$

（对于任意的 ${}_1x_i^1$，${}_1y_i^1$ 而言，${}_1x_i^1, {}_1y_i^1 \in z$，${}_1x_i^1$，${}_1y_i^1$ 同奇偶

，•••,

对于任意的 ${}_1x_i^{n_1}$，${}_1y_i^{n_1}$ 而言，${}_1x_i^{n_1}, {}_1y_i^{n_1} \in z$，${}_1x_i^{n_1}$，${}_1y_i^{n_1}$ 同奇偶）

注：对于这些集合 $\left\{{}_1x_1^1,\cdots,{}_1x_{n_0}^1,{}_1y_1^1,\cdots,{}_1y_{n_0}^1\right\}$，••••••，集合 $\left\{{}_1x_1^{n_1},\cdots,{}_1x_{n_0}^{n_1},{}_1y_1^{n_1},\cdots,{}_1y_{n_0}^{n_1}\right\}$ 之间的任意两个集合均不相同。

第一次一级，二级，三级作图的三级作图后，平面上所存在（A1，B1）为:

$$\left(\frac{x_1}{2}\left(\frac{{}_1^1A}{2}-\sqrt{\frac{1}{{}_1^1A^2+{}_1^1B^2}-\frac{1}{4}}\bullet {}_1^1B\right)-\frac{\sqrt{3}y_1}{2}\left(\frac{{}_1^1B}{2}+\sqrt{\frac{1}{{}_1^1A^2+B_1^2}-\frac{1}{4}}\bullet {}_1^1A\right),\right.$$

$$\left.\frac{x_1}{2}\left(\frac{{}_1^1B}{2}+\sqrt{\frac{1}{{}_1^1A^2+{}_1^1B^2}-\frac{1}{4}}\bullet {}_1^1A\right)+\frac{\sqrt{3}y_1}{2}\left(\frac{{}_1^1A}{2}-\sqrt{\frac{1}{{}_1^1A^2+{}_1^1B^2}-\frac{1}{4}}\bullet {}_1^1B\right)\right)$$ 和

$$\left(\frac{x_1}{2}\left(\frac{{}_1^1A}{2}+\sqrt{\frac{1}{{}_1^1A^2+{}_1^1B^2}-\frac{1}{4}}\bullet {}_1^1B\right)-\frac{\sqrt{3}y_1}{2}\left(\frac{{}_1^1B}{2}-\sqrt{\frac{1}{{}_1^1A^2+{}_1^1B^2}-\frac{1}{4}}\bullet {}_1^1A\right),\right.$$

$$\left.\frac{x_1}{2}\left(\frac{{}_1^1A}{2}-\sqrt{\frac{1}{{}_1^1A^2+{}_1^1B^2}-\frac{1}{4}}\bullet {}_1^1A\right)+\frac{\sqrt{3}y_1}{2}\left(\frac{{}_1^1A}{2}+\sqrt{\frac{1}{{}_1^1A^2+{}_1^1B^2}-\frac{1}{4}}\bullet {}_1^1B\right)\right)$$

…………

$$(\frac{x_1}{2}(\frac{{}_1^{n_1}A}{2}-\sqrt{\frac{1}{{}_1^{n_1}A^2+{}_1^{n_1}B^2}-\frac{1}{4}}\bullet {}_1^{n_1}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}_1^{n_1}B}{2}+\sqrt{\frac{1}{{}_1^{n_1}A^2+{}_1^{n_1}B^2}-\frac{1}{4}}\bullet {}_1^{n_1}A),$$

$$\frac{x_1}{2}(\frac{{}_1^{n_1}B}{2}+\sqrt{\frac{1}{{}_1^{n_1}A^2+{}_1^{n_1}B^2}-\frac{1}{4}}\bullet {}_1^{n_1}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}_1^{n_1}A}{2}-\sqrt{\frac{1}{{}_1^{n_1}A^2+{}_1^{n_1}B^2}-\frac{1}{4}}\bullet {}_1^{n_1}B))\text{和}$$

$$(\frac{x_1}{2}(\frac{{}_1^{n_1}A}{2}+\sqrt{\frac{1}{{}_1^{n_1}A^2+{}_1^{n_1}B^2}-\frac{1}{4}}\bullet {}_1^{n_1}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}_1^{n_1}B}{2}-\sqrt{\frac{1}{{}_1^{n_1}A^2+{}_1^{n_1}B^2}-\frac{1}{4}}\bullet {}_1^{n_1}A),$$

$$\frac{x_1}{2}(\frac{{}_1^{n_1}B}{2}-\sqrt{\frac{1}{{}_1^{n_1}A^2+{}_1^{n_1}B^2}-\frac{1}{4}}\bullet {}_1^{n_1}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}_1^{n_1}A}{2}+\sqrt{\frac{1}{{}_1^{n_1}A^2+{}_1^{n_1}B^2}-\frac{1}{4}}\bullet {}_1^{n_1}B))$$

$(x_1\in\{2,1,-1\}$ ， $y_1\in\{1,0,-1\}$ ， 且 x_1，y_1 的取值满足

$$(\frac{x_1}{2}(\frac{{}_1^{x}A}{2}-\sqrt{\frac{1}{{}_1^{x}A^2+{}_1^{x}B^2}-\frac{1}{4}}\bullet {}_1^{x}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}_1^{x}B}{2}+\sqrt{\frac{1}{{}_1^{x}A^2+{}_1^{x}B^2}-\frac{1}{4}}\bullet {}_1^{x}A))^2+$$

$$(\frac{x_1}{2}(\frac{{}_1^{x}B}{2}+\sqrt{\frac{1}{{}_1^{x}A^2+{}_1^{x}B^2}-\frac{1}{4}}\bullet {}_1^{x}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}_1^{x}A}{2}-\sqrt{\frac{1}{{}_1^{x}A^2+{}_1^{x}B^2}-\frac{1}{4}}\bullet {}_1^{x}B))^2=1)$$

=> 第二次一级、二级、三级作图的二级作图使得平面上的点的集合为：

$$(\sum_{i=1}^{n_1}\begin{array}{l}\frac{x_{2i-1}}{2}(\frac{{}_1^{i}A}{2}-\sqrt{\frac{1}{{}_1^{i}A^2+{}_1^{i}B^2}-\frac{1}{4}}\bullet {}_1^{i}B)-\frac{\sqrt{3}y_{2i-1}}{2}(\frac{{}_1^{i}B}{2}+\sqrt{\frac{1}{{}_1^{i}A^2+{}_1^{i}B^2}-\frac{1}{4}}\bullet {}_1^{i}A)+\\ \frac{x_{2i}}{2}(\frac{{}_1^{i}A}{2}+\sqrt{\frac{1}{{}_1^{i}A^2+{}_1^{i}B^2}-\frac{1}{4}}\bullet {}_1^{i}B))-\frac{\sqrt{3}y_{2i}}{2}(\frac{{}_1^{i}B}{2}-\sqrt{\frac{1}{{}_1^{i}A^2+{}_1^{i}B^2}-\frac{1}{4}}\bullet {}_1^{i}A)\end{array},$$

$$\sum_{i=1}^{n_1}\begin{array}{l}\frac{x_{2i-1}}{2}(\frac{{}_1^{i}B}{2}+\sqrt{\frac{1}{{}_1^{i}A^2+{}_1^{i}B^2}-\frac{1}{4}}\bullet {}_1^{i}A)+\frac{\sqrt{3}y_{2i-1}}{2}(\frac{{}_1^{i}A}{2}-\sqrt{\frac{1}{{}_1^{i}A^2+{}_1^{i}B^2}-\frac{1}{4}}\bullet {}_1^{i}B)+\\ \frac{x_{2i}}{2}(\frac{{}_1^{i}B}{2}-\sqrt{\frac{1}{{}_1^{i}A^2+{}_1^{i}B^2}-\frac{1}{4}}\bullet {}_1^{i}A)+\frac{\sqrt{3}y_{2i}}{2}(\frac{{}_1^{i}A}{2}+\sqrt{\frac{1}{{}_1^{i}A^2+{}_1^{i}B^2}-\frac{1}{4}}\bullet {}_1^{i}B)\end{array})$$

（对于任意的 $x_{2i-1},y_{2i-1},x_{2i},y_{2i}$ 而言，$x_{2i-1},y_{2i-1},x_{2i},y_{2i}\in z$，$x_{2i-1},y_{2i-1}$ 同奇偶，x_{2i},y_{2i} 同奇偶）

化简得：

$$(\sum_{i=1}^{n_1}(\frac{(x_{2i-1}+x_{2i})_1^iA-\sqrt{3}(y_{2i-1}+y_{2i})_1^iB}{4}-\sqrt{\frac{1}{{}_1^iA^2+{}_1^iB^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})_1^iB+\sqrt{3}(y_{2i-1}-y_{2i})_1^iA}{2})),$$

$$\sum_{i=1}^{n_1}(\frac{(x_{2i-1}+x_{2i})_1^iB+\sqrt{3}(y_{2i-1}+y_{2i})_1^iA}{4}+\sqrt{\frac{1}{{}_1^iA^2+{}_1^iB^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})_1^iA-\sqrt{3}(y_{2i-1}-y_{2i})_1^iB}{2})))$$ 。

=>满足第 $m(m \in N^+, 2 \le m < n)$ 次一级、二级、三级作图的三级作图两点坐标差为：

$$(\sum_{i=1}^{n_{m-1}}(\frac{(x_{2i-1}+x_{2i})_{m-1}^{\ i}A-\sqrt{3}(y_{2i-1}+y_{2i})_{m-1}^{\ i}B}{4}-\sqrt{\frac{1}{{}_{m-1}^{\ i}A^2+{}_{m-1}^{\ i}B^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})_{m-1}^{\ i}B+\sqrt{3}(y_{2i-1}-y_{2i})_{m-1}^{\ i}A}{2})),$$

$$\sum_{i=1}^{n_{m-1}}(\frac{(x_{2i-1}+x_{2i})_{m-1}^{\ i}B+\sqrt{3}(y_{2i-1}+y_{2i})_{m-1}^{\ i}A}{4}+\sqrt{\frac{1}{{}_{m-1}^{\ i}A^2+{}_{m-1}^{\ i}B^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})_{m-1}^{\ i}A-\sqrt{3}(y_{2i-1}-y_{2i})_{m-1}^{\ i}B}{2})))$$

设横坐标差为 ${}_mA$，纵坐标差为 ${}_mB$。则第 m 次一级、二级、三级作图的三级作图所确定（A1，B1）通式为：

$$(\frac{{}_mA}{2}+\sqrt{\frac{1}{{}_mA^2+{}_mB^2}-\frac{1}{4}}\bullet{}_mB,\frac{{}_mB}{2}-\sqrt{\frac{1}{{}_mA^2+{}_mB^2}-\frac{1}{4}}\bullet{}_mA)\text{和}$$

$$(\frac{{}_mA}{2}-\sqrt{\frac{1}{{}_mA^2+{}_mB^2}-\frac{1}{4}}\bullet{}_mB,\frac{{}_mB}{2}+\sqrt{\frac{1}{{}_mA^2+{}_mB^2}-\frac{1}{4}}\bullet{}_mA)$$

$$
\begin{aligned}
&{}_mA^2+{}_mB^2=\\
&\quad\frac{(x_{2i-1}+x_{2i})\bullet(x_{2j-1}+x_{2j})+3(y_{2i-1}+y_{2i})\bullet(y_{2j-1}+y_{2j})}{4\times 2}-\\
&\quad\frac{\sqrt{3}((x_{2i-1}+x_{2i})\bullet(y_{2j-1}-y_{2j})-(y_{2i-1}+y_{2i})\bullet(x_{2j-1}-x_{2j}))}{4\times 2}\bullet\sqrt{\frac{1}{{}_{m-1}^{\ \ j}A^2+{}_{m-1}^{\ \ j}B^2}-\frac{1}{4}}+\\
&\sum_{1\le i,j\le n_{m-1}}\frac{\sqrt{3}((x_{2j-1}+x_{2j})\bullet(y_{2i-1}-y_{2i})-(y_{2j-1}+y_{2j})\bullet(x_{2i-1}-x_{2i}))}{4\times 2}\bullet\sqrt{\frac{1}{{}_{m-1}^{\ \ i}A^2+{}_{m-1}^{\ \ i}B^2}-\frac{1}{4}}+\\
&\quad\frac{((x_{2i-1}-x_{2i})\bullet(x_{2j-1}-x_{2j})+3(y_{2i-1}-y_{2i})\bullet(y_{2j-1}-y_{2j}))}{4}\\
&\quad\bullet\sqrt{\frac{1}{{}_{m-1}^{\ \ i}A^2+{}_{m-1}^{\ \ i}B^2}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_{m-1}^{\ \ j}A^2+{}_{m-1}^{\ \ j}B^2}-\frac{1}{4}})({}_{m-1}^{\ \ i}A\,{}_{m-1}^{\ \ j}A+{}_{m-1}^{\ \ i}B\,{}_{m-1}^{\ \ j}B)\\
&\quad\frac{\sqrt{3}((x_{2i-1}+x_{2i})\bullet(y_{2j-1}+y_{2j})-(x_{2j-1}+x_{2j})\bullet(y_{2i-1}+y_{2i}))}{4\times 4}+\\
&\quad\frac{((x_{2i-1}+x_{2i})\bullet(x_{2j-1}-x_{2j})+3(y_{2i-1}+y_{2i})\bullet(y_{2j-1}-y_{2j}))}{4\times 2}\bullet\sqrt{\frac{1}{{}_{m-1}^{\ \ j}A^2+{}_{m-1}^{\ \ j}B^2}-\frac{1}{4}}-\\
&-\sum_{1\le i,j\le n_{m-1}}\frac{((x_{2j-1}+x_{2j})\bullet(x_{2i-1}-x_{2i})+3(y_{2j-1}+y_{2j})\bullet(y_{2i-1}-y_{2i}))}{4\times 2}\bullet\sqrt{\frac{1}{{}_{m-1}^{\ \ i}A^2+{}_{m-1}^{\ \ i}B^2}-\frac{1}{4}}-\\
&\quad\frac{\sqrt{3}((y_{2i-1}-y_{2i})\bullet(x_{2j-1}-x_{2j})-(y_{2j-1}-y_{2j})\bullet(x_{2i-1}-x_{2i}))}{4}\\
&\quad\bullet\sqrt{\frac{1}{{}_{m-1}^{\ \ i}A^2+{}_{m-1}^{\ \ i}B^2}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_{m-1}^{\ \ j}A^2+{}_{m-1}^{\ \ j}B^2}-\frac{1}{4}})({}_{m-1}^{\ \ i}A\,{}_{m-1}^{\ \ j}B-{}_{m-1}^{\ \ i}B\,{}_{m-1}^{\ \ j}A)
\end{aligned}
$$

(对于任意的 $x_{2i-1},y_{2i-1},x_{2i},y_{2i},x_{2j-1},y_{2j-1},x_{2j},y_{2j}$ 而言，$x_{2i-1},y_{2i-1},x_{2i},y_{2i},x_{2j-1},y_{2j-1},x_{2j},y_{2j}\in z,x_{2i-1},y_{2i-1}$ 同奇偶，x_{2i},y_{2i} 同奇偶，x_{2j-1},y_{2j-1} 同奇偶，x_{2j},y_{2j} 同奇偶)

化简后，得：

$$
{}_{m}A^2+{}_{m}B^2=\sum_{i=1}^{n_{m-1}}\frac{x_{2i-1}x_{2i}+3y_{2i-1}y_{2i}}{4}({}_{m-1}^{\;\,i}A^2+{}_{m-1}^{\;\,i}B^2)+\frac{(x_{2i-1}-x_{2i})^2+3(y_{2i-1}-y_{2i})^2}{4}+\frac{\sqrt{3}(x_{2i-1}y_{2i}-x_{2i}y_{2i-1})}{2}\sqrt{4-({}_{m-1}^{\;\,i}A^2+{}_{m-1}^{\;\,i}B^2)}\bullet\sqrt{{}_{m-1}^{\;\,i}A^2+{}_{m-1}^{\;\,i}B^2}
$$

$$
\frac{(x_{2i-1}+x_{2i})\bullet(x_{2j-1}+x_{2j})+3(y_{2i-1}+y_{2i})\bullet(y_{2j-1}+y_{2j})}{4\times 2}-
$$

$$
\frac{\sqrt{3}((x_{2i-1}+x_{2i})\bullet(y_{2j-1}-y_{2j})-(y_{2i-1}+y_{2i})\bullet(x_{2j-1}-x_{2j}))}{4}\bullet\sqrt{\frac{1}{{}_{m-1}^{\;\,j}A^2+{}_{m-1}^{\;\,j}B^2}-\frac{1}{4}}+
$$

$$
+\sum_{1\le i<j\le n_{m-1}}\frac{\sqrt{3}((x_{2j-1}+x_{2j})\bullet(y_{2i-1}-y_{2i})-(y_{2j-1}+y_{2j})\bullet(x_{2i-1}-x_{2i}))}{4}\bullet\sqrt{\frac{1}{{}_{m-1}^{\;\,i}A^2+{}_{m-1}^{\;\,i}B^2}-\frac{1}{4}}+
$$

$$
\frac{((x_{2i-1}-x_{2i})\bullet(x_{2j-1}-x_{2j})+3(y_{2i-1}-y_{2i})\bullet(y_{2j-1}-y_{2j}))}{2}
$$

$$
\bullet\sqrt{\frac{1}{{}_{m-1}^{\;\,i}A^2+{}_{m-1}^{\;\,i}B^2}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_{m-1}^{\;\,j}A^2+{}_{m-1}^{\;\,j}B^2}-\frac{1}{4}})({}_{m-1}^{\;\,i}A\,{}_{m-1}^{\;\,j}A+{}_{m-1}^{\;\,i}B\,{}_{m-1}^{\;\,j}B)
$$

$$
\frac{\sqrt{3}((x_{2i-1}+x_{2i})\bullet(y_{2j-1}+y_{2j})-(x_{2j-1}+x_{2j})\bullet(y_{2i-1}+y_{2i}))}{4\times 2}+
$$

$$
\frac{((x_{2i-1}+x_{2i})\bullet(x_{2j-1}-x_{2j})+3(y_{2i-1}+y_{2i})\bullet(y_{2j-1}-y_{2j}))}{4}\bullet\sqrt{\frac{1}{{}_{m-1}^{\;\,j}A^2+{}_{m-1}^{\;\,j}B^2}-\frac{1}{4}}-
$$

$$
-\sum_{1\le i<j\le n_{m-1}}\frac{((x_{2j-1}+x_{2j})\bullet(x_{2i-1}-x_{2i})+3(y_{2j-1}+y_{2j})\bullet(y_{2i-1}-y_{2i}))}{4}\bullet\sqrt{\frac{1}{{}_{m-1}^{\;\,i}A^2+{}_{m-1}^{\;\,i}B^2}-\frac{1}{4}}-
$$

$$
\frac{\sqrt{3}((y_{2i-1}-y_{2i})\bullet(x_{2j-1}-x_{2j})-(y_{2j-1}-y_{2j})\bullet(x_{2i-1}-x_{2i}))}{2}
$$

$$
\bullet\sqrt{\frac{1}{{}_{m-1}^{\;\,i}A^2+{}_{m-1}^{\;\,i}B^2}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_{m-1}^{\;\,j}A^2+{}_{m-1}^{\;\,j}B^2}-\frac{1}{4}})({}_{m-1}^{\;\,i}A\,{}_{m-1}^{\;\,j}B-{}_{m-1}^{\;\,i}B\,{}_{m-1}^{\;\,j}A)
$$

将第 m 次一级，二级，三级作图后，平面上所存在（A1，B1）的均化为

$$\left(\frac{{}_mA}{2}+\sqrt{\frac{1}{{}_mA^2+{}_mB^2}-\frac{1}{4}}\bullet {}_mB,\ \frac{{}_mB}{2}-\sqrt{\frac{1}{{}_mA^2+{}_mB^2}-\frac{1}{4}}\bullet {}_mA\right)$$和

$$\left(\frac{{}_mA}{2}-\sqrt{\frac{1}{{}_mA^2+{}_mB^2}-\frac{1}{4}}\bullet {}_mB,\ \frac{{}_mB}{2}+\sqrt{\frac{1}{{}_mA^2+{}_mB^2}-\frac{1}{4}}\bullet {}_mA\right)$$

(${}_mA$ =

$$\sum_{i=1}^{n_{m-1}}\left(\frac{(x_{2i-1}+x_{2i})\,{}_{m-1}^{i}A-\sqrt{3}(y_{2i-1}+y_{2i})\,{}_{m-1}^{i}B}{4}-\sqrt{\frac{1}{{}_{m-1}^{i}A^2+{}_{m-1}^{i}B^2}-\frac{1}{4}}\bullet\left(\frac{(x_{2i-1}-x_{2i})\,{}_{m-1}^{i}B+\sqrt{3}(y_{2i-1}-y_{2i})\,{}_{m-1}^{i}A}{2}\right)\right)$$

,

${}_mB$ =

$$\sum_{i=1}^{n_{m-1}}\left(\frac{(x_{2i-1}+x_{2i})\,{}_{m-1}^{i}B+\sqrt{3}(y_{2i-1}+y_{2i})\,{}_{m-1}^{i}A}{4}+\sqrt{\frac{1}{{}_{m-1}^{i}A^2+{}_{m-1}^{i}B^2}-\frac{1}{4}}\bullet\left(\frac{(x_{2i-1}-x_{2i})\,{}_{m-1}^{i}A-\sqrt{3}(y_{2i-1}-y_{2i})\,{}_{m-1}^{i}B}{2}\right)\right)$$

)这样的形式。设

$${}_m^1A=\sum_{i=1}^{n_{m-1}}\left(\frac{({}_mx_{2i-1}^1+{}_mx_{2i}^1)\,{}_{m-1}^{i}A-\sqrt{3}({}_my_{2i-1}^1+{}_my_{2i}^1)\,{}_{m-1}^{i}B}{4}-\sqrt{\frac{1}{{}_{m-1}^{i}A^2+{}_{m-1}^{i}B^2}-\frac{1}{4}}\bullet\left(\frac{({}_mx_{2i-1}^1-{}_mx_{2i}^1)\,{}_{m-1}^{i}B+\sqrt{3}({}_my_{2i-1}^1-{}_my_{2i}^1)\,{}_{m-1}^{i}A}{2}\right)\right),$$

$${}_m^1B=\sum_{i=1}^{n_{m-1}}\left(\frac{({}_mx_{2i-1}^1+{}_mx_{2i}^1)\,{}_{m-1}^{i}B+\sqrt{3}({}_my_{2i-1}^1+{}_my_{2i}^1)\,{}_{m-1}^{i}A}{4}+\sqrt{\frac{1}{{}_{m-1}^{i}A^2+{}_{m-1}^{i}B^2}-\frac{1}{4}}\bullet\left(\frac{({}_mx_{2i-1}^1-{}_nx_{2i}^1)\,{}_{n-1}^{i}A-\sqrt{3}({}_my_{2i-1}^1-{}_my_{2i}^1)\,{}_{m-1}^{i}B}{2}\right)\right).$$

…………

$${}_{m}^{n_m}A=\sum_{i=1}^{n_{m-1}}(\frac{({}_m x_{2i-1}^{n_m}+{}_m x_{2i}^{n_m})\,{}_{m-1}^{\ \ i}A-\sqrt{3}({}_m y_{2i-1}^{n_m}+{}_m y_{2i}^{n_m})\,{}_{m-1}^{\ \ i}B}{4}-\sqrt{\frac{1}{{}_{m-1}^{\ \ i}A^2+{}_{m-1}^{\ \ i}B^2}-\frac{1}{4}}\bullet(\frac{({}_m x_{2i-1}^{n_m}-{}_n x_{2i}^{n_m})\,{}_{m-1}^{\ \ i}B+\sqrt{3}({}_m y_{2i-1}^{n_m}-{}_m y_{2i}^{n_m})\,{}_{m-1}^{\ \ i}A}{2})),$$

$${}_{m}^{n_m}B=\sum_{i=1}^{n_{m-1}}(\frac{({}_m x_{2i-1}^{n_m}+{}_m x_{2i}^{n_m})\,{}_{m-1}^{\ \ i}B+\sqrt{3}({}_m y_{2i-1}^{n_m}+{}_m y_{2i}^{n_m})\,{}_{m-1}^{\ \ i}A}{4}+\sqrt{\frac{1}{{}_{m-1}^{\ \ i}A^2+{}_{m-1}^{\ \ i}B^2}-\frac{1}{4}}\bullet(\frac{({}_m x_{2i-1}^{n_m}-{}_m x_{2i}^{n_m})\,{}_{m-1}^{\ \ i}A-\sqrt{3}({}_m y_{2i-1}^{n_m}-{}_m y_{2i}^{n_m})\,{}_{m-1}^{\ \ i}B}{2})).$$

（对于任意的 ${}_m x_{2i-1}^1, {}_m y_{2i-1}^1, {}_m x_{2i}^1, {}_m y_{2i}^1$ 而言，${}_m x_{2i-1}^1, {}_m y_{2i-1}^1, {}_m x_{2i}^1, {}_m y_{2i}^1 \in z, {}_m x_{2i-1}^1, {}_m y_{2i-1}^1$ 同奇偶，${}_m x_{2i}^1, {}_m y_{2i}^1$ 同奇偶

，•••,

对于任意的 ${}_m x_{2i-1}^{n_m}, {}_m y_{2i-1}^{n_m}, {}_m x_{2i}^{n_m}, {}_m y_{2i}^{n_m}$ 而言，${}_m x_{2i-1}^{n_m}, {}_m y_{2i-1}^{n_m}, {}_m x_{2i}^{n_m}, {}_m y_{2i}^{n_m} \in z, {}_m x_{2i-1}^{n_m}, {}_m y_{2i-1}^{n_m}$ 同奇偶，${}_m x_{2i}^{n_m}, {}_m y_{2i}^{n_m}$ 同奇偶）

注：对于这些集合 $\left\{{}_m x_1^1,\cdots,{}_m x_{2n_{m-1}}^1,{}_m y_1^1,\cdots,{}_m y_{2n_{m-1}}^1\right\}$，•••••••，集合 $\left\{{}_m x_1^{n_m},\cdots,{}_m x_{2n_{m-1}}^{n_m},{}_m y_1^{n_m},\cdots,{}_m y_{2n_{m-1}}^{n_m}\right\}$ 之间的任意两个集合均不相同。

=>第 m 次一级，二级，三级作图的三级作图后，平面上所存在（A1，B1）为:

$$(\frac{x_1}{2}(\frac{{}_m^1A}{2}-\sqrt{\frac{1}{{}_m^1A^2+{}_m^1B^2}-\frac{1}{4}}\bullet{}_m^1B)-\frac{\sqrt{3}y_1}{2}(\frac{{}_m^1B}{2}+\sqrt{\frac{1}{{}_m^1A^2+{}_m^1B^2}-\frac{1}{4}}\bullet{}_m^1A),$$

$$\frac{x_1}{2}(\frac{{}_m^1B}{2}+\sqrt{\frac{1}{{}_m^1A^2+{}_m^1B^2}-\frac{1}{4}}\bullet{}_m^1A)+\frac{\sqrt{3}y_1}{2}(\frac{{}_m^1A}{2}-\sqrt{\frac{1}{{}_m^1A^2+{}_m^1B^2}-\frac{1}{4}}\bullet{}_m^1B))\text{和}$$

$$(\frac{x_1}{2}(\frac{{}_m^1A}{2}+\sqrt{\frac{1}{{}_m^1A^2+{}_m^1B^2}-\frac{1}{4}}\bullet{}_m^1B)-\frac{\sqrt{3}y_1}{2}(\frac{{}_m^1B}{2}-\sqrt{\frac{1}{{}_m^1A^2+{}_m^1B^2}-\frac{1}{4}}\bullet{}_m^1A),$$

$$\frac{x_1}{2}(\frac{{}_m^1A}{2}-\sqrt{\frac{1}{{}_m^1A^2+{}_m^1B^2}-\frac{1}{4}}\bullet{}_m^1A)+\frac{\sqrt{3}y_1}{2}(\frac{{}_m^1A}{2}+\sqrt{\frac{1}{{}_m^1A^2+{}_m^1B^2}-\frac{1}{4}}\bullet{}_m^1B))$$

…………

$$(\frac{x_1}{2}(\frac{{}^{n_m}_{m}A}{2}-\sqrt{\frac{1}{{}^{n_m}_{m}A^2+{}^{n_m}_{m}B^2}-\frac{1}{4}}\bullet {}^{n_m}_{m}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}^{n_m}_{m}B}{2}+\sqrt{\frac{1}{{}^{n_m}_{m}A^2+{}^{n_m}_{m}B^2}-\frac{1}{4}}\bullet {}^{n_m}_{m}A),$$

$$\frac{x_1}{2}(\frac{{}^{n_m}_{m}B}{2}+\sqrt{\frac{1}{{}^{n_m}_{m}A^2+{}^{n_m}_{m}B^2}-\frac{1}{4}}\bullet {}^{n_m}_{m}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}^{n_m}_{m}A}{2}-\sqrt{\frac{1}{{}^{n_m}_{m}A^2+{}^{n_m}_{m}B^2}-\frac{1}{4}}\bullet {}^{n_m}_{m}B))\text{和}$$

$$(\frac{x_1}{2}(\frac{{}^{n_m}_{m}A}{2}+\sqrt{\frac{1}{{}^{n_m}_{m}A^2+{}^{n_m}_{m}B^2}-\frac{1}{4}}\bullet {}^{n_m}_{m}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}^{n_m}_{m}B}{2}-\sqrt{\frac{1}{{}^{n_m}_{m}A^2+{}^{n_m}_{m}B^2}-\frac{1}{4}}\bullet {}^{n_m}_{m}A),$$

$$\frac{x_1}{2}(\frac{{}^{n_m}_{m}B}{2}-\sqrt{\frac{1}{{}^{n_m}_{m}A^2+{}^{n_m}_{m}B^2}-\frac{1}{4}}\bullet {}^{n_m}_{m}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}^{n_m}_{m}A}{2}+\sqrt{\frac{1}{{}^{n_m}_{m}A^2+{}^{n_m}_{m}B^2}-\frac{1}{4}}\bullet {}^{n_m}_{m}B))$$

$(x_1\in\{2,1,-1\}$ ， $y_1\in\{1,0,-1\}$ ， 且 x_1, y_1 的取值满足

$$(\frac{x_1}{2}(\frac{{}^{x}_{m}A}{2}-\sqrt{\frac{1}{{}^{x}_{m}A^2+{}^{x}_{m}B^2}-\frac{1}{4}}\bullet {}^{x}_{n}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}^{x}_{m}B}{2}+\sqrt{\frac{1}{{}^{x}_{m}A^2+{}^{x}_{m}B^2}-\frac{1}{4}}\bullet {}^{x}_{m}A))^2+$$

$$(\frac{x_1}{2}(\frac{{}^{x}_{m}B}{2}+\sqrt{\frac{1}{{}^{x}_{m}A^2+{}^{x}_{m}B^2}-\frac{1}{4}}\bullet {}^{x}_{n}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}^{x}_{m}A}{2}-\sqrt{\frac{1}{{}^{x}_{m}A^2+{}^{x}_{m}B^2}-\frac{1}{4}}\bullet {}^{x}_{m}B))^2=1)$$

=>第 $m+1$ 次一级、二级、三级作图的二级作图，使得平面上的点的集合为：

$$(\sum_{i=1}^{n_m}\begin{matrix}(\frac{x_{2i-1}}{2}(\frac{{}^{i}_{m}A}{2}-\sqrt{\frac{1}{{}^{i}_{m}A^2+{}^{i}_{m}B^2}-\frac{1}{4}}\bullet {}^{i}_{m}B)-\frac{\sqrt{3}y_{2i-1}}{2}(\frac{{}^{i}_{m}B}{2}+\sqrt{\frac{1}{{}^{i}_{m}A^2+{}^{i}_{m}B^2}-\frac{1}{4}}\bullet {}^{i}_{m}A)+\\ \frac{x_{2i}}{2}(\frac{{}^{i}_{m}A}{2}+\sqrt{\frac{1}{{}^{i}_{m}A^2+{}^{i}_{m}B^2}-\frac{1}{4}}\bullet {}^{i}_{m}B))-\frac{\sqrt{3}y_{2i}}{2}(\frac{{}^{i}_{m}B}{2}-\sqrt{\frac{1}{{}^{i}_{m}A^2+{}^{i}_{m}B^2}-\frac{1}{4}}\bullet {}^{i}_{m}A))\end{matrix},$$

$$\sum_{i=1}^{n_m}\begin{matrix}(\frac{x_{2i-1}}{2}(\frac{{}^{i}_{m}B}{2}+\sqrt{\frac{1}{{}^{i}_{m}A^2+{}^{i}_{m}B^2}-\frac{1}{4}}\bullet {}^{i}_{m}A)+\frac{\sqrt{3}y_{2i-1}}{2}(\frac{{}^{i}_{m}A}{2}-\sqrt{\frac{1}{{}^{i}_{m}A^2+{}^{i}_{m}B^2}-\frac{1}{4}}\bullet {}^{i}_{m}B)+\\ \frac{x_{2i}}{2}(\frac{{}^{i}_{m}B}{2}-\sqrt{\frac{1}{{}^{i}_{m}A^2+{}^{i}_{m}B^2}-\frac{1}{4}}\bullet {}^{i}_{m}A)+\frac{\sqrt{3}y_{2i}}{2}(\frac{{}^{i}_{m}A}{2}+\sqrt{\frac{1}{{}^{i}_{m}A^2+{}^{i}_{m}B^2}-\frac{1}{4}}\bullet {}^{i}_{m}B))\end{matrix})$$

（对于任意的 $x_{2i-1},y_{2i-1},x_{2i},y_{2i}$ 而言，$x_{2i-1},y_{2i-1},x_{2i},y_{2i}\in z$ ，x_{2i-1},y_{2i-1} 同奇偶， x_{2i},y_{2i} 同奇偶）

化简得：

$$(\sum_{i=1}^{n_m}(\frac{(x_{2i-1}+x_{2i})_m^iA-\sqrt{3}(y_{2i-1}+y_{2i})_m^iB}{4}-\sqrt{\frac{1}{{}_m^iA^2+{}_m^iB^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})_m^iB+\sqrt{3}(y_{2i-1}-y_{2i})_m^iA}{2})),$$

$$\sum_{i=1}^{n_m}(\frac{(x_{2i-1}+x_{2i})_m^iB+\sqrt{3}(y_{2i-1}+y_{2i})_m^iA}{4}+\sqrt{\frac{1}{{}_m^iA^2+{}_m^iB^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})_m^iA-\sqrt{3}(y_{2i-1}-y_{2i})_m^iB}{2})))$$ 。

=> 满 足 第 n 次 一 级 、 二 级 、 三 级 作 图 的 三 级 作 图 两 点 坐 标 差 为 ：

$$(\sum_{i=1}^{n_{n-1}}(\frac{(x_{2i-1}+x_{2i})_{n-1}^iA-\sqrt{3}(y_{2i-1}+y_{2i})_{n-1}^iB}{4}-\sqrt{\frac{1}{{}_{n-1}^iA^2+{}_{n-1}^iB^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})_{n-1}^iB+\sqrt{3}(y_{2i-1}-y_{2i})_{n-1}^iA}{2})),$$

$$\sum_{i=1}^{n_{n-1}}(\frac{(x_{2i-1}+x_{2i})_{n-1}^iB+\sqrt{3}(y_{2i-1}+y_{2i})_{n-1}^iA}{4}+\sqrt{\frac{1}{{}_{n-1}^iA^2+{}_{n-1}^iB^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})_{n-1}^iA-\sqrt{3}(y_{2i-1}-y_{2i})_{n-1}^iB}{2})))$$

设横坐标差为 ${}_nA$，纵坐标差为 ${}_nB$。则第 n 次一级、二级、三级作图的三级作图所确定（A1，B1）通式为：

$$(\frac{{}_nA}{2}+\sqrt{\frac{1}{{}_nA^2+{}_nB^2}-\frac{1}{4}}\bullet{}_nB,\frac{{}_nB}{2}-\sqrt{\frac{1}{{}_nA^2+{}_nB^2}-\frac{1}{4}}\bullet{}_nA)\text{和}$$

$$(\frac{{}_nA}{2}-\sqrt{\frac{1}{{}_nA^2+{}_nB^2}-\frac{1}{4}}\bullet{}_nB,\frac{{}_nB}{2}+\sqrt{\frac{1}{{}_nA^2+{}_nB^2}-\frac{1}{4}}\bullet{}_nA)$$

$$
{}_{n}A^2+{}_{n}B^2=
$$

$$
\frac{(x_{2i-1}+x_{2i})\bullet(x_{2j-1}+x_{2j})+3(y_{2i-1}+y_{2i})\bullet(y_{2j-1}+y_{2j})}{4\times 2}-
$$

$$
\frac{\sqrt{3}((x_{2i-1}+x_{2i})\bullet(y_{2j-1}-y_{2j})-(y_{2i-1}+y_{2i})\bullet(x_{2j-1}-x_{2j}))}{4\times 2}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ j}A^2+{}_{n-1}^{\ j}B^2}-\frac{1}{4}}+
$$

$$
\sum_{1\le i,\,j\le n_{n-1}}\frac{\sqrt{3}((x_{2j-1}+x_{2j})\bullet(y_{2i-1}-y_{2i})-(y_{2j-1}+y_{2j})\bullet(x_{2i-1}-x_{2i}))}{4\times 2}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ i}A^2+{}_{n-1}^{\ i}B^2}-\frac{1}{4}}+
$$

$$
\frac{((x_{2i-1}-x_{2i})\bullet(x_{2j-1}-x_{2j})+3(y_{2i-1}-y_{2i})\bullet(y_{2j-1}-y_{2j}))}{4}
$$

$$
\bullet\sqrt{\frac{1}{{}_{n-1}^{\ i}A^2+{}_{n-1}^{\ i}B^2}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ j}A^2+{}_{n-1}^{\ j}B^2}-\frac{1}{4}})({}_{n-1}^{\ i}A\,{}_{n-1}^{\ j}A+{}_{n-1}^{\ i}B\,{}_{n-1}^{\ j}B)
$$

$$
\frac{\sqrt{3}((x_{2i-1}+x_{2i})\bullet(y_{2j-1}+y_{2j})-(x_{2j-1}+x_{2j})\bullet(y_{2i-1}+y_{2i}))}{4\times 4}+
$$

$$
\frac{((x_{2i-1}+x_{2i})\bullet(x_{2j-1}-x_{2j})+3(y_{2i-1}+y_{2i})\bullet(y_{2j-1}-y_{2j}))}{4\times 2}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ j}A^2+{}_{n-1}^{\ j}B^2}-\frac{1}{4}}-
$$

$$
-\sum_{1\le i,\,j\le n_{n-1}}\frac{((x_{2j-1}+x_{2j})\bullet(x_{2i-1}-x_{2i})+3(y_{2j-1}+y_{2j})\bullet(y_{2i-1}-y_{2i}))}{4\times 2}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ i}A^2+{}_{n-1}^{\ i}B^2}-\frac{1}{4}}-
$$

$$
\frac{\sqrt{3}((y_{2i-1}-y_{2i})\bullet(x_{2j-1}-x_{2j})-(y_{2j-1}-y_{2j})\bullet(x_{2i-1}-x_{2i}))}{4}
$$

$$
\bullet\sqrt{\frac{1}{{}_{n-1}^{\ i}A^2+{}_{n-1}^{\ i}B^2}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ j}A^2+{}_{n-1}^{\ j}B^2}-\frac{1}{4}})({}_{n-1}^{\ i}A\,{}_{n-1}^{\ j}B-{}_{n-1}^{\ i}B\,{}_{n-1}^{\ j}A)
$$

(对于任意的 $x_{2i-1},y_{2i-1},x_{2i},y_{2i},x_{2j-1},y_{2j-1},x_{2j},y_{2j}$ 而言，$x_{2i-1},y_{2i-1},x_{2i},y_{2i},x_{2j-1},y_{2j-1},x_{2j},y_{2j}\in z,x_{2i-1},y_{2i-1}$ 同奇偶，x_{2i},y_{2i} 同奇偶，x_{2j-1},y_{2j-1} 同奇偶，x_{2j},y_{2j} 同奇偶)

化简后，得：

$$
{}_{n}A^{2}+{}_{n}B^{2}=\sum_{i=1}^{n_{n-1}}\begin{array}{l}\dfrac{x_{2i-1}x_{2i}+3y_{2i-1}y_{2i}}{4}({}_{n-1}^{\ \ i}A^{2}+{}_{n-1}^{\ \ i}B^{2})+\dfrac{(x_{2i-1}-x_{2i})^{2}+3(y_{2i-1}-y_{2i})^{2}}{4}\\ +\dfrac{\sqrt{3}(x_{2i-1}y_{2i}-x_{2i}y_{2i-1})}{2}\sqrt{4-({}_{n-1}^{\ \ i}A^{2}+{}_{n-1}^{\ \ i}B^{2})}\bullet\sqrt{{}_{n-1}^{\ \ i}A^{2}+{}_{n-1}^{\ \ i}B^{2}}\end{array}
$$

$$
\begin{aligned}
&\frac{(x_{2i-1}+x_{2i})\bullet(x_{2j-1}+x_{2j})+3(y_{2i-1}+y_{2i})\bullet(y_{2j-1}+y_{2j})}{4\times 2}-\\
&\frac{\sqrt{3}((x_{2i-1}+x_{2i})\bullet(y_{2j-1}-y_{2j})-(y_{2i-1}+y_{2i})\bullet(x_{2j-1}-x_{2j}))}{4}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ \ j}A^{2}+{}_{n-1}^{\ \ j}B^{2}}-\frac{1}{4}}+\\
+\sum_{1\le i<j\le n_{n-1}}&\frac{\sqrt{3}((x_{2j-1}+x_{2j})\bullet(y_{2i-1}-y_{2i})-(y_{2j-1}+y_{2j})\bullet(x_{2i-1}-x_{2i}))}{4}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ \ i}A^{2}+{}_{n-1}^{\ \ i}B^{2}}-\frac{1}{4}}+\\
&\frac{((x_{2i-1}-x_{2i})\bullet(x_{2j-1}-x_{2j})+3(y_{2i-1}-y_{2i})\bullet(y_{2j-1}-y_{2j}))}{2}\\
&\bullet\sqrt{\frac{1}{{}_{n-1}^{\ \ i}A^{2}+{}_{n-1}^{\ \ i}B^{2}}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ \ j}A^{2}+{}_{n-1}^{\ \ j}B^{2}}-\frac{1}{4}})({}_{n-1}^{\ \ i}A\,{}_{n-1}^{\ \ j}A+{}_{n-1}^{\ \ i}B\,{}_{n-1}^{\ \ j}B)\\
&\frac{\sqrt{3}((x_{2i-1}+x_{2i})\bullet(y_{2j-1}+y_{2j})-(x_{2j-1}+x_{2j})\bullet(y_{2i-1}+y_{2i}))}{4\times 2}+\\
&\frac{((x_{2i-1}+x_{2i})\bullet(x_{2j-1}-x_{2j})+3(y_{2i-1}+y_{2i})\bullet(y_{2j-1}-y_{2j}))}{4}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ \ j}A^{2}+{}_{n-1}^{\ \ j}B^{2}}-\frac{1}{4}}-\\
-\sum_{1\le i<j\le n_{n-1}}&\frac{((x_{2j-1}+x_{2j})\bullet(x_{2i-1}-x_{2i})+3(y_{2j-1}+y_{2j})\bullet(y_{2i-1}-y_{2i}))}{4}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ \ i}A^{2}+{}_{n-1}^{\ \ i}B^{2}}-\frac{1}{4}}-\\
&\frac{\sqrt{3}((y_{2i-1}-y_{2i})\bullet(x_{2j-1}-x_{2j})-(y_{2j-1}-y_{2j})\bullet(x_{2i-1}-x_{2i}))}{2}\\
&\bullet\sqrt{\frac{1}{{}_{n-1}^{\ \ i}A^{2}+{}_{n-1}^{\ \ i}B^{2}}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_{n-1}^{\ \ j}A^{2}+{}_{n-1}^{\ \ j}B^{2}}-\frac{1}{4}})({}_{n-1}^{\ \ i}A\,{}_{n-1}^{\ \ j}B-{}_{n-1}^{\ \ i}B\,{}_{n-1}^{\ \ j}A)
\end{aligned}
$$

将第 n 次一级，二级，三级作图后，平面上所存在（A1，B1）的均化为

$$(\frac{{}_nA}{2}+\sqrt{\frac{1}{{}_nA^2+{}_nB^2}-\frac{1}{4}}\bullet{}_nB,\frac{{}_nB}{2}-\sqrt{\frac{1}{{}_nA^2+{}_nB^2}-\frac{1}{4}}\bullet{}_nA)$$ 和

$$(\frac{{}_nA}{2}-\sqrt{\frac{1}{{}_nA^2+{}_nB^2}-\frac{1}{4}}\bullet{}_nB,\frac{{}_nB}{2}+\sqrt{\frac{1}{{}_nA^2+{}_nB^2}-\frac{1}{4}}\bullet{}_nA)$$

$$({}_nA =$$

$$\sum_{i=1}^{n_{n-1}}(\frac{(x_{2i-1}+x_{2i})\,{}_{n-1}^{\;i}A-\sqrt{3}(y_{2i-1}+y_{2i})\,{}_{n-1}^{\;i}B}{4}-\sqrt{\frac{1}{{}_{n-1}^{\;i}A^2+{}_{n-1}^{\;i}B^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})\,{}_{n-1}^{\;i}B+\sqrt{3}(y_{2i-1}-y_{2i})\,{}_{n-1}^{\;i}A}{2})),$$

$${}_nB =$$

$$\sum_{i=1}^{n_{n-1}}(\frac{(x_{2i-1}+x_{2i})\,{}_{n-1}^{\;i}B+\sqrt{3}(y_{2i-1}+y_{2i})\,{}_{n-1}^{\;i}A}{4}+\sqrt{\frac{1}{{}_{n-1}^{\;i}A^2+{}_{n-1}^{\;i}B^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i})\,{}_{n-1}^{\;i}A-\sqrt{3}(y_{2i-1}-y_{2i})\,{}_{n-1}^{\;i}B}{2})))$$

这样的形式。设：

$${}_n^1A=\sum_{i=1}^{n_{n-1}}(\frac{({}_nx_{2i-1}^1+{}_nx_{2i}^1)\,{}_{n-1}^{\;i}A-\sqrt{3}({}_ny_{2i-1}^1+{}_ny_{2i}^1)\,{}_{n-1}^{\;i}B}{4}-\sqrt{\frac{1}{{}_{n-1}^{\;i}A^2+{}_{n-1}^{\;i}B^2}-\frac{1}{4}}\bullet(\frac{({}_nx_{2i-1}^1-{}_nx_{2i}^1)\,{}_{n-1}^{\;i}B+\sqrt{3}({}_ny_{2i-1}^1-{}_ny_{2i}^1)\,{}_{n-1}^{\;i}A}{2})),$$

$${}_n^1B=\sum_{i=1}^{n_{n-1}}(\frac{({}_nx_{2i-1}^1+{}_nx_{2i}^1)\,{}_{n-1}^{\;i}B+\sqrt{3}({}_ny_{2i-1}^1+{}_ny_{2i}^1)\,{}_{n-1}^{\;i}A}{4}+\sqrt{\frac{1}{{}_{n-1}^{\;i}A^2+{}_{n-1}^{\;i}B^2}-\frac{1}{4}}\bullet(\frac{({}_nx_{2i-1}^1-{}_nx_{2i}^1)\,{}_{n-1}^{\;i}A-\sqrt{3}({}_ny_{2i-1}^1-{}_ny_{2i}^1)\,{}_{n-1}^{\;i}B}{2})).$$

…………

$${}_n^{n_n}A=\sum_{i=1}^{n_{n-1}}(\frac{({}_nx_{2i-1}^{n_n}+{}_nx_{2i}^{n_n})\,{}_{n-1}^{\;i}A-\sqrt{3}({}_ny_{2i-1}^{n_n}+{}_ny_{2i}^{n_n})\,{}_{n-1}^{\;i}B}{4}-\sqrt{\frac{1}{{}_{n-1}^{\;i}A^2+{}_{n-1}^{\;i}B^2}-\frac{1}{4}}\bullet(\frac{({}_nx_{2i-1}^{n_n}-{}_nx_{2i}^{n_n})\,{}_{n-1}^{\;i}B+\sqrt{3}({}_ny_{2i-1}^{n_n}-{}_ny_{2i}^{n_n})\,{}_{n-1}^{\;i}A}{2})),$$

$$ {}_{n}^{n_n}B=\sum_{i=1}^{n_{n-1}}\begin{pmatrix}\dfrac{({}_{n}x_{2i-1}^{n_n}+{}_{n}x_{2i}^{n_n})\,{}_{n-1}^{\ \ i}B+\sqrt{3}({}_{n}y_{2i-1}^{n_n}+{}_{n}y_{2i}^{n_n})\,{}_{n-1}^{\ \ i}A}{4}+\\ \sqrt{\dfrac{1}{{}_{n-1}^{\ \ i}A^2+{}_{n-1}^{\ \ i}B^2}-\dfrac{1}{4}}\bullet(\dfrac{({}_{n}x_{2i-1}^{n_n}-{}_{n}x_{2i}^{n_n})\,{}_{n-1}^{\ \ i}A-\sqrt{3}({}_{n}y_{2i-1}^{n_n}-{}_{n}y_{2i}^{n_n})\,{}_{n-1}^{\ \ i}B}{2})\end{pmatrix}. $$

（对于任意的 ${}_{n}x_{2i-1}^{1}, {}_{n}y_{2i-1}^{1}, {}_{n}x_{2i}^{1}, {}_{n}y_{2i}^{1}$ 而言，${}_{n}x_{2i-1}^{1}, {}_{n}y_{2i-1}^{1}, {}_{n}x_{2i}^{1}, {}_{n}y_{2i}^{1}\in z$, ${}_{n}x_{2i-1}^{1}, {}_{n}y_{2i-1}^{1}$ 同奇偶，${}_{n}x_{2i}^{1}, {}_{n}y_{2i}^{1}$ 同奇偶

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对于任意的 ${}_{n}x_{2i-1}^{n_n}, {}_{n}y_{2i-1}^{n_n}, {}_{n}x_{2i}^{n_n}, {}_{n}y_{2i}^{n_n}$ 而言，${}_{n}x_{2i-1}^{n_n}, {}_{n}y_{2i-1}^{n_n}, {}_{n}x_{2i}^{n_n}, {}_{n}y_{2i}^{n_n}\in z$, ${}_{n}x_{2i-1}^{n_n}, {}_{n}y_{2i-1}^{n_n}$ 同奇偶，${}_{n}x_{2i}^{n_n}, {}_{n}y_{2i}^{n_n}$ 同奇偶）

注：对于这些集合 $\left\{{}_{n}x_{1}^{1},\cdots,{}_{n}x_{2n_{n-1}}^{1},{}_{n}y_{1}^{1},\cdots,{}_{n}y_{2n_{n-1}}^{1}\right\}$，•••••••，集合 $\left\{{}_{n}x_{1}^{n_n},\cdots,{}_{n}x_{2n_{n-1}}^{n_n},{}_{n}y_{1}^{n_n},\cdots,{}_{n}y_{2n_{n-1}}^{n_n}\right\}$ 之间的任意两个集合均不相同。

第 n 次一级，二级，三级作图的三级作图后，平面上所存在（A1，B1）为:

$$ \Bigg(\frac{x_1}{2}\Big(\frac{{}_{n}^{1}A}{2}-\sqrt{\frac{1}{{}_{n}^{1}A^2+{}_{n}^{1}B^2}-\frac{1}{4}}\bullet{}_{n}^{1}B\Big)-\frac{\sqrt{3}y_1}{2}\Big(\frac{{}_{n}^{1}B}{2}+\sqrt{\frac{1}{{}_{n}^{1}A^2+{}_{n}^{1}B^2}-\frac{1}{4}}\bullet{}_{n}^{1}A\Big), $$

$$ \frac{x_1}{2}\Big(\frac{{}_{n}^{1}B}{2}+\sqrt{\frac{1}{{}_{n}^{1}A^2+{}_{n}^{1}B^2}-\frac{1}{4}}\bullet{}_{n}^{1}A\Big)+\frac{\sqrt{3}y_1}{2}\Big(\frac{{}_{n}^{1}A}{2}-\sqrt{\frac{1}{{}_{n}^{1}A^2+{}_{n}^{1}B^2}-\frac{1}{4}}\bullet{}_{n}^{1}B\Big)\Bigg) \text{和} $$

$$ \Bigg(\frac{x_1}{2}\Big(\frac{{}_{n}^{1}A}{2}+\sqrt{\frac{1}{{}_{n}^{1}A^2+{}_{n}^{1}B^2}-\frac{1}{4}}\bullet{}_{n}^{1}B\Big)-\frac{\sqrt{3}y_1}{2}\Big(\frac{{}_{n}^{1}B}{2}-\sqrt{\frac{1}{{}_{n}^{1}A^2+{}_{n}^{1}B^2}-\frac{1}{4}}\bullet{}_{n}^{1}A\Big), $$

$$ \frac{x_1}{2}\Big(\frac{{}_{n}^{1}A}{2}-\sqrt{\frac{1}{{}_{n}^{1}A^2+{}_{n}^{1}B^2}-\frac{1}{4}}\bullet{}_{n}^{1}A\Big)+\frac{\sqrt{3}y_1}{2}\Big(\frac{{}_{n}^{1}A}{2}+\sqrt{\frac{1}{{}_{n}^{1}A^2+{}_{n}^{1}B^2}-\frac{1}{4}}\bullet{}_{n}^{1}B\Big)\Bigg) $$

…………

$$(\frac{x_1}{2}(\frac{{}_n^{n_n}A}{2}-\sqrt{\frac{1}{{}_n^{n_n}A^2+{}_n^{n_n}B^2}-\frac{1}{4}}\bullet{}_n^{n_n}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}_n^{n_n}B}{2}+\sqrt{\frac{1}{{}_n^{n_n}A^2+{}_n^{n_n}B^2}-\frac{1}{4}}\bullet{}_n^{n_n}A),$$

$$\frac{x_1}{2}(\frac{{}_n^{n_n}B}{2}+\sqrt{\frac{1}{{}_n^{n_n}A^2+{}_n^{n_n}B^2}-\frac{1}{4}}\bullet{}_n^{n_n}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}_n^{n_n}A}{2}-\sqrt{\frac{1}{{}_n^{n_n}A^2+{}_n^{n_n}B^2}-\frac{1}{4}}\bullet{}_n^{n_n}B))\text{和}$$

$$(\frac{x_1}{2}(\frac{{}_n^{n_n}A}{2}+\sqrt{\frac{1}{{}_n^{n_n}A^2+{}_n^{n_n}B^2}-\frac{1}{4}}\bullet{}_n^{n_n}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}_n^{n_n}B}{2}-\sqrt{\frac{1}{{}_n^{n_n}A^2+{}_n^{n_n}B^2}-\frac{1}{4}}\bullet{}_n^{n_n}A),$$

$$\frac{x_1}{2}(\frac{{}_n^{n_n}B}{2}-\sqrt{\frac{1}{{}_n^{n_n}A^2+{}_n^{n_n}B^2}-\frac{1}{4}}\bullet{}_n^{n_n}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}_n^{n_n}A}{2}+\sqrt{\frac{1}{{}_n^{n_n}A^2+{}_n^{n_n}B^2}-\frac{1}{4}}\bullet{}_n^{n_n}B))$$

$(x_1\in\{2,1,-1\}$，$y_1\in\{1,0,-1\}$，且 x_1，y_1 的取值满足

$$(\frac{x_1}{2}(\frac{{}_n^{x}A}{2}-\sqrt{\frac{1}{{}_n^{x}A^2+{}_n^{x}B^2}-\frac{1}{4}}\bullet{}_n^{x}B)-\frac{\sqrt{3}y_1}{2}(\frac{{}_n^{x}B}{2}+\sqrt{\frac{1}{{}_n^{x}A^2+{}_n^{x}B^2}-\frac{1}{4}}\bullet{}_n^{x}A))^2+$$

$$(\frac{x_1}{2}(\frac{{}_n^{x}B}{2}+\sqrt{\frac{1}{{}_n^{x}A^2+{}_n^{x}B^2}-\frac{1}{4}}\bullet{}_n^{x}A)+\frac{\sqrt{3}y_1}{2}(\frac{{}_n^{x}A}{2}-\sqrt{\frac{1}{{}_n^{x}A^2+{}_n^{x}B^2}-\frac{1}{4}}\bullet{}_n^{x}B))^2=1)$$

=>第 n+1（n>0）次一级、二级、三级作图的二级作图，使得平面上的点的集合为：

$$(\sum_{i=1}^{n_n}\begin{matrix}(\frac{x_{2i-1}}{2}(\frac{{}_n^{i}A}{2}-\sqrt{\frac{1}{{}_n^{i}A^2+{}_n^{i}B^2}-\frac{1}{4}}\bullet{}_n^{i}B)-\frac{\sqrt{3}y_{2i-1}}{2}(\frac{{}_n^{i}B}{2}+\sqrt{\frac{1}{{}_n^{i}A^2+{}_n^{i}B^2}-\frac{1}{4}}\bullet{}_n^{i}A)+\\ \frac{x_{2i}}{2}(\frac{{}_n^{i}A}{2}+\sqrt{\frac{1}{{}_n^{i}A^2+{}_n^{i}B^2}-\frac{1}{4}}\bullet{}_n^{i}B))-\frac{\sqrt{3}y_{2i}}{2}(\frac{{}_n^{i}B}{2}-\sqrt{\frac{1}{{}_n^{i}A^2+{}_n^{i}B^2}-\frac{1}{4}}\bullet{}_n^{i}A))\end{matrix},$$

$$\sum_{i=1}^{n_n}\begin{matrix}(\frac{x_{2i-1}}{2}(\frac{{}_n^{i}B}{2}+\sqrt{\frac{1}{{}_n^{i}A^2+{}_n^{i}B^2}-\frac{1}{4}}\bullet{}_n^{i}A)+\frac{\sqrt{3}y_{2i-1}}{2}(\frac{{}_n^{i}A}{2}-\sqrt{\frac{1}{{}_n^{i}A^2+{}_n^{i}B^2}-\frac{1}{4}}\bullet{}_n^{i}B)+\\ \frac{x_{2i}}{2}(\frac{{}_n^{i}B}{2}-\sqrt{\frac{1}{{}_n^{i}A^2+{}_n^{i}B^2}-\frac{1}{4}}\bullet{}_n^{i}A)+\frac{\sqrt{3}y_{2i}}{2}(\frac{{}_n^{i}A}{2}+\sqrt{\frac{1}{{}_n^{i}A^2+{}_n^{i}B^2}-\frac{1}{4}}\bullet{}_n^{i}B))\end{matrix})$$

（对于任意的 $x_{2i-1},y_{2i-1},x_{2i},y_{2i}$ 而言，$x_{2i-1},y_{2i-1},x_{2i},y_{2i}\in z$，$x_{2i-1},y_{2i-1}$ 同奇偶，x_{2i},y_{2i} 同奇偶）

化简得：

$$(\sum_{i=1}^{n_n}(\frac{(x_{2i-1}+x_{2i}){}_{n}^{i}A-\sqrt{3}(y_{2i-1}+y_{2i}){}_{n}^{i}B}{4}-\sqrt{\frac{1}{{}_{n}^{i}A^2+{}_{n}^{i}B^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i}){}_{n}^{i}B+\sqrt{3}(y_{2i-1}-y_{2i}){}_{n}^{i}A}{2})),$$

$$\sum_{i=1}^{n_n}(\frac{(x_{2i-1}+x_{2i}){}_{n}^{i}B+\sqrt{3}(y_{2i-1}+y_{2i}){}_{n}^{i}A}{4}+\sqrt{\frac{1}{{}_{n}^{i}A^2+{}_{n}^{i}B^2}-\frac{1}{4}}\bullet(\frac{(x_{2i-1}-x_{2i}){}_{n}^{i}A-\sqrt{3}(y_{2i-1}-y_{2i}){}_{n}^{i}B}{2})))$$ 。

四、生锈圆规作图方程的结构及其的一些特点：

第一、因子及其分类，因子积链及其分类和其它规定：

(通过以上内容，规定（**注：除了设和特别说明外，下面的内容的设定与上面的内容的设定一样**)：

1、 **特别说明：**(n, i $\in N^+, n>i$)，$(i_i$取遍$1,\cdots,n_i,0\le i\le n)$。

将 $\sqrt{\frac{1}{{}_{i}^{i_i}A^2+{}_{i}^{i_i}B^2}-\frac{1}{4}}$ 定义为$\sqrt{\frac{1}{l^2}-\frac{1}{4}}$型因子、将 $\frac{(x_{2i_n-1}+x_{2i_n})}{4}$，$\frac{({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}})}{4}$，$\frac{{}_{1}x_{i_0}^{i_1}}{2}$，$\cos\theta_{i_0}$ 定义为 x_1 型因子、将 $\frac{(x_{2i_n-1}-x_{2i_n})}{2}$，$\frac{({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2}$ 定义为 x_2 型因子、将 $-\frac{(x_{2i_n-1}-x_{2i_n})}{2}$，$-\frac{({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2}$ 定义为 $-x_2$ 型因子、将 $\frac{\sqrt{\frac{1}{{}_{n}^{i_n}A^2+{}_{n}^{i_n}B^2}-\frac{1}{4}}\bullet(x_{2i_n-1}-x_{2i_n})}{2}$，$\frac{\sqrt{\frac{1}{{}_{i}^{i_i}A^2+{}_{i}^{i_i}B^2}-\frac{1}{4}}\bullet({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2}$ 定义为 $x_2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子、将 $-\frac{\sqrt{\frac{1}{{}_{n}^{i_n}A^2+{}_{n}^{i_n}B^2}-\frac{1}{4}}\bullet(x_{2i_n-1}-x_{2i_n})}{2}$，$-\frac{\sqrt{\frac{1}{{}_{i}^{i_i}A^2+{}_{i}^{i_i}B^2}-\frac{1}{4}}\bullet({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2}$ 定义为 $-x_2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子：将 $\frac{\sqrt{3}(y_{2i_n-1}+y_{2i_n})}{4}$，

$\frac{\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}})}{4}$，$\frac{\sqrt{3}{}_1y_{i_0}^{i_1}}{2}$，$\sin\theta_{i_0}$ 定义为 y_1 型因子、将 $-\frac{\sqrt{3}(y_{2i_n-1}+y_{2i_n})}{4}$，$-\frac{\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}})}{4}$，$-\frac{\sqrt{3}{}_1y_{i_0}^{i_1}}{2}$ 定义为 $-y_1$ 型因子、将 $-\frac{\sqrt{3}(y_{2i_n-1}-y_{2i_n})}{2}$，$-\frac{\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}-{}_{i+1}y_{2i_i}^{i_{i+1}})}{2}$ 定义为 $-y_2$ 型因子　、将 $-\frac{\sqrt{\frac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\frac{1}{4}}\bullet\sqrt{3}(y_{2i_n-1}-y_{2i_n})}{2}$，$-\frac{\sqrt{\frac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\frac{1}{4}}\bullet\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}-{}_{i+1}y_{2i_i}^{i_{i+1}})}{2}$ 定义为 $-y_2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子　。将 $-\frac{\sqrt{\frac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\frac{1}{4}}\bullet\sqrt{3}(y_{2i_n-1}-y_{2i_n})}{2}$，$-\frac{\sqrt{\frac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\frac{1}{4}}\bullet\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}-{}_{i+1}y_{2i_i}^{i_{i+1}})}{2}$ 定义为 $-y_2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子　。x_1 型因子、y_1 型因子、$-y_1$ 型因子、统称为“1”型因子。x_2 型因子、$-x_2$ 型因子、$-y_2$ 型因子、统称为“2”型因子。将 $\sqrt{\frac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\frac{1}{4}}$ 定义为第 n 级 $\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子、将 $\sqrt{\frac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\frac{1}{4}}$ 定义为第 i 级 $\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子。$\frac{(x_{2i_n-1}+x_{2i_n})}{4}$ 定义为第 n+1 级 x_1 型因子、$\frac{\sqrt{\frac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\frac{1}{4}}\bullet(x_{2i_n-1}-x_{2i_n})}{2}$ 定义为第 n+1 级 $x_2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子、$-\frac{\sqrt{\frac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\frac{1}{4}}\bullet(x_{2i_n-1}-x_{2i_n})}{2}$ 定义为第 n+1 级 $-x_2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子、将

$\dfrac{({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}})}{4}$定义为第 i+1 级$x_1$型因子、$\dfrac{\sqrt{\dfrac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\dfrac{1}{4}}\bullet({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2}$定义为第 i+1 级$x_2\bullet\sqrt{\dfrac{1}{l^2}-\dfrac{1}{4}}$型因子、$-\dfrac{\sqrt{\dfrac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\dfrac{1}{4}}\bullet({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2}$定义为第 i+1 级$-x_2\bullet\sqrt{\dfrac{1}{l^2}-\dfrac{1}{4}}$型因子，将$\dfrac{{}_1x_{i_0}^{i_1}}{2}$定义为第 1 级$x_1$型因子，将$\cos\theta_{i_0}$定义为第 0 级$x_1$型因子。

同理：将$\dfrac{\sqrt{3}(y_{2i_n-1}+y_{2i_n})}{4}$定义为第 n+1 级$y_1$型因子、将$-\dfrac{\sqrt{3}(y_{2i_n-1}+y_{2i_n})}{4}$定义为第 n+1 级$-y_1$型因子、将$-\dfrac{\sqrt{\dfrac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\dfrac{1}{4}}\bullet\sqrt{3}(y_{2i_n-1}-y_{2i_n})}{2}$定义为第 n+1 级$-y_2\bullet\sqrt{\dfrac{1}{l^2}-\dfrac{1}{4}}$型因子，将$\dfrac{\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}})}{4}$定义为第 i+1 级$y_1$型因子、将$-\dfrac{\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}})}{4}$定义为第 i+1 级$-y_1$型因子、将$-\dfrac{\sqrt{\dfrac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\dfrac{1}{4}}\bullet\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}-{}_{i+1}y_{2i_i}^{i_{i+1}})}{2}$定义为第 i+1 级$-y_2\bullet\sqrt{\dfrac{1}{l^2}-\dfrac{1}{4}}$型因子，将$\dfrac{\sqrt{3}\,{}_1y_{i_0}^{i_1}}{2}$定义为第 1 级$y_1$型因子、将$-\dfrac{\sqrt{3}\,{}_1y_{i_0}^{i_1}}{2}$定义为第 1 级$-y_1$型因子、将$\sin\theta_{i_0}$定义为第 0 级$y_1$型因子。第 n+1 级$x_1$型因子、第 n+1 级$y_1$型因子、第 n+1 级$-y_1$型因子统称为第 n+1 级“1”型因子，第 i+1 级x_1型因子、第 i+1 级y_1型因子、第 i+1 级$-y_1$型因子统称为第 i+1 级“1”型因子，第+1 级x_1型因子、第 1 级y_1型因子、第 1 级$-y_1$型因子统称为第 1 级“1”型因子。第 n+1 级x_2型因子、第 n+1 级$-x_2$型因子、第 n+1 级$-y_2$型因子统称为

第 n+1 级“ 2 ”型因子，第 i+1 级 x_2 型因子、第 i+1 级 $-x_2$ 型因子、第 i+1 级 $-y_2$ 型因子统称为第 i+1 级“ 2 ”型因子。第 n+1 级 $x_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子、第 n+1 级 $-x_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子、第 n+1 级 $-y_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子统称为第 n+1 级“ $2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ ”型因子，第 i+1 级 $x_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子、第 i+1 级 $-x_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子、第 i+1 级 $-y_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子统称为第 i+1 级“ $2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ ”型因子。“1”型因子、“ 2 ”型因子和“ $2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ ”型因子统称为因子;“1”型因子和“ $2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ ”型因子统称为整体因子。称 “ 2 ”型因子为“ $2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ ”型因子的有理体。称“ $\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ ”为“ $2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ ”型因子的无理体。

因子的数积：规定：所有的 x_1 型因子的数积是 1，所有的 y_1 型因子和 $-y_1$ 型因子的数积是 2，所有的 x_2 型因子和 $-x_2$ 型因子的数积是 2，所有的 $-y_2$ 型因子的数积是 3，所有的 $\sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子的数积是 0，所有的 $x_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子和 $-x_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子的数积是 2，所有的 $-y_2 \bullet \sqrt{\frac{1}{l^2}-\frac{1}{4}}$ 型因子的数积是 3。

规定集合 $1_{i_n}=\left\{\frac{(x_{2i_n-1}+x_{2i_n})}{4},\frac{\sqrt{3}(y_{2i_n-1}+y_{2i_n})}{4},-\frac{\sqrt{3}(y_{2i_n-1}+y_{2i_n})}{4}\right\}$ 、集合

$$1_{i_i}^{i_{i+1}}=\left\{\frac{({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}})}{4},\frac{\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}})}{4},-\frac{\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}})}{4}\right\}$$ 、集合

$$1_{i_0}^{i_1}=\left\{\frac{{}_1x_{i_0}^{i_1}}{2},\frac{\sqrt{3}\,{}_1y_{i_0}^{i_1}}{2},-\frac{\sqrt{3}\,{}_1y_{i_0}^{i_1}}{2}\right\}$$ 、集合

$$2_{i_n}=\left\{\frac{(x_{2i_n-1}-x_{2i_n})}{2},-\frac{(x_{2i_n-1}-x_{2i_n})}{2},-\frac{\sqrt{3}(y_{2i_n-1}-y_{2i_n})}{2}\right\}$$ 、集合

$$2_{i_i}^{i_{i+1}}=\left\{\frac{({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2},-\frac{({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2},-\frac{\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}-{}_{i+1}y_{2i_i}^{i_{i+1}})}{2}\right\}$$ 、集合

$$2_{i_n}\sqrt{\frac{1}{(l_n^{i_n})^2}-\frac{1}{4}}=\left\{\frac{\sqrt{\frac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\frac{1}{4}}\bullet(x_{2i_n-1}-x_{2i_n})}{2},\ -\frac{\sqrt{\frac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\frac{1}{4}}\bullet(x_{2i_n-1}-x_{2i_n})}{2},\ -\frac{\sqrt{\frac{1}{{}_n^{i_n}A^2+{}_n^{i_n}B^2}-\frac{1}{4}}\bullet\sqrt{3}(y_{2i_n-1}-y_{2i_n})}{2}\right\}$$ 、集合

$$2_{i_i}^{i_{i+1}}\bullet\sqrt{\frac{1}{(l_i^{i_i})^2}-\frac{1}{4}}=\left\{\frac{\sqrt{\frac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\frac{1}{4}}\bullet({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2},\ -\frac{\sqrt{\frac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\frac{1}{4}}\bullet({}_{i+1}x_{2i_i-1}^{i_{i+1}}-{}_{i+1}x_{2i_i}^{i_{i+1}})}{2},\ -\frac{\sqrt{\frac{1}{{}_i^{i_i}A^2+{}_i^{i_i}B^2}-\frac{1}{4}}\bullet\sqrt{3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}-{}_{i+1}y_{2i_i}^{i_{i+1}})}{2}\right\}$$ 。

若一种含有 y-x+1 个元素的集合，且这 y-x+1 个元素满足两个条件，条件一：分别是一个第 y 级因子、一个第 y-1 级因子、•••、一个第 x 级因子,：条件二：第 y 级因子是集合 $1_{i_{y-1}}^{i_y}\bigcup 2_{i_{y-1}}^{i_y}\bullet\sqrt{\frac{1}{(l_y^{i_y})^2}-\frac{1}{4}}$ 中的元素（当 y=n+1 时，为集合 $1_{i_n}\bigcup 2_{i_n}\bullet\sqrt{\frac{1}{(l_n^{i_n})^2}-\frac{1}{4}}$ 中的元素）、第 y-1 级因子是集合 $1_{i_{y-2}}^{i_{y-1}}\bigcup 2_{i_{y-2}}^{i_{y-1}}\bullet\sqrt{\frac{1}{(l_{y-2}^{i_{y-1}})^2}-\frac{1}{4}}$ 中的元素、•••、第 x 级因子是集合 $1_{i_{x-1}}^{i_x}\bigcup 2_{i_{x-1}}^{i_x}\bullet\sqrt{\frac{1}{(l_x^{i_x})^2}-\frac{1}{4}}$ 或集合中的元素。则称这种集合是一个连环因子集合。

2、**特别说明**：$(i\in N, i\le n+1)$。第 i 级"1"型因子，第"$2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$"型因子统称为第 i 级整体因子。

3、因子积链的定义：当 $x>1, x\in N^+$ 时：x 个因子相乘的表达式是一个因子积链，称这 x 个因子是这个因子积链的基因子，x 是这个因子积链的基因子数。整体因子积链的定义：当 $x>1, x\in N^+$ 时：x 个整体因子相乘的表达式是一个整体因子积链，称这 x 个整体因子是这个整体因子积链的整体基因子，x 是这个整体因子积链的整体基因子数。（注："1" 型因子、"2"型因子和"$\sqrt{\frac{1}{l^2}-\frac{1}{4}}$"型因子是基因子，"1" 型因子和"$2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$"型因子是整体基因子。整体因子积链的基因子是指"1" 型因子、"2"型因子和"$\sqrt{\frac{1}{l^2}-\frac{1}{4}}$"型因子，整体因子积链的整体基因子是指"1" 型因子和"$2\bullet\sqrt{\frac{1}{l^2}-\frac{1}{4}}$"型因子。

由因子积链的定义可知：当 $x>1, x\in N^+$ 时：x 个因子积链相乘的表达式也是一个因子积链，所以称这 x 个因子积链是这个因子积链的基因子积链，这 x 个因子积链的基因子数之和是这个因子积链的基因子。当 $x>1, x\in N^+$ 时：x 个整体因子积链相乘的表达式也是一

个整体因子积链，所以称这 x 个整体因子积链是这个整体因子积链的整体基因子积链，这 x 个整体因子积链的基因子数之和是这个整体因子积链的基因子数。

c_1 个第 x_1 级整体因子 $\times$ c_1 个第 x_1-1 级整体因子 $\times\cdots\cdots\times$ c_1 个第 x_2+1 级整体因子 $\times$ c_2 个第 x_2 级整体因子 $\times$ c_2 个第 x_2-1 级整体因子 $\times\cdots\cdots\times$ c_2 个第 x_3+1 级整体因子 $\times\cdots\cdots\cdots\cdots\cdots\cdots\times$ c_x 个第 x_x 级整体因子 $\times$ c_x 个第 x_x-1 级整体因子 $\times\cdots\cdots\times$ c_x 个第 $x_{x+1}+1$ 级整体因子 $(c_1、\cdots\cdots、c_x \in N, x_1、\cdots\cdots、x_{x+1} \in N, x_1 > \cdots > x_{x+1}+1)$ 为一个 ${}^{c_1}x_1 \to {}^{c_1}(x_2+1) \to {}^{c_2}x_2 \to {}^{c_2}(x_3+1) \to \cdots\cdots\cdots \to {}^{c_x}x_x \to {}^{c_x}(x_{x+1}+1)$ 型整体因子积链；一个 ${}^{c_1}x_1 \to {}^{c_1}(x_2+1) \to {}^{c_2}x_2 \to {}^{c_2}(x_3+1) \to \cdots\cdots\cdots \to {}^{c_x}x_x \to {}^{c_x}(x_{x+1}+1)$ 型整体因子积链 $\times$ 一个 ${}^{d_1}y_1 \to {}^{d_1}(y_2+1) \to {}^{d_2}x_2 \to {}^{d_2}(x_3+1) \to \cdots\cdots\cdots \to {}^{d_y}y_x \to {}^{d_y}(y_{x+1}+1)$ 型整体因子积链 $(c_1、\cdots\cdots、c_x、d_1、\cdots\cdots、d_x \in N, x_1、y_1、\cdots\cdots、x_{x+1}、y_{x+1} \in N, x_1 > y_1 > x_2 \cdots > y_x > x_{x+1} > y_{x+1})$ 为一个 ${}^{c_1}x_1 \to {}^{c_1+d_1}y_1 \to {}^{c_1+d_1}(x_2+1) \to {}^{c_2+d_1}x_2 \to {}^{c_2+d_1}(y_2+1) \to \cdots\cdots\cdots \to {}^{c_x+d_{x-1}}x_x \to {}^{c_x+d_{x-1}}(y_x+1) \to {}^{c_x+d_x}(y_x) \to {}^{c_x+d_x}(x_{x+1}+1) \to {}^{d_x}(y_{x+1}+1)$ 型整体因子积链。

连环整体因子积链：一个整体因子积链所有的整体基因子是一个连环因子集合的全部元素或多个连环因子集合的全部元素时，则称这个整体因子积链为一个连环整体因子积链，这个 ${}^{c_1}x_1 \to {}^{c_1}(x_2+1) \to {}^{c_2}x_2 \to {}^{c_2}(x_3+1) \to \cdots\cdots\cdots \to {}^{c_x}x_x \to {}^{c_x}(x_{x+1}+1)$ 型整体因子积链为一个 ${}^{c_1}x_1 \to {}^{c_1}(x_2+1) \to {}^{c_2}x_2 \to {}^{c_2}(x_3+1) \to \cdots\cdots\cdots \to {}^{c_x}x_x \to {}^{c_x}(x_{x+1}+1)$ 型连环整体因子积链。

一个整体因子积链的基因子中所有“1”型因子和所有的“2”型因子相乘的表达式是这个整体因子积链的有理体。整体因子积链的基因子中所有的“$\sqrt{\frac{1}{l^2}-\frac{1}{4}}$”型因子相乘的表达式是这个整体因子积链的无理体。

因子积链的数积：规定：一个因子积链的所有基因子的数积之和是这个因子积链数积，一般用符号 a 表示。

因子积链的符号：只有一个整体因子积链的所有基因子中的$-y_1$型因子、$-x_2$型因子和$-y_2$型因子的总数为一个奇数时，这个整体因子积链的符号为-1；只有一个整体因子积链的所有基因子中的$-y_1$型因子、$-x_2$型因子和$-y_2$型因子的总数为一个偶数时，这个整体因子积链的符号为 1。

（**特别说明：** $y, x \in N, y > x$。）${}^1y \to {}^1x$ 型整体因子积链的第 y 级（整体）基因子是这个 ${}^1y \to {}^1x$ 型整体因子积链的首项（整体）基因子，第 x 级（整体）基因子是这个 ${}^1y \to {}^1x$ 型整体因子积链的尾项（整体）基因子。

第二、F_n 的表达式：

利用方法一级，二级，三级作图构造生锈圆规作图方程的内容可推得：${}_x^{x_i}A\,(1 \le x \le n-1, 1 \le x_i \le n_x)$ 的表达式和 F_n 的横坐标表达式分别是所有满足因子积链的数积 a，$a \ge x+1$ 并与 $x+1$ 同奇偶，且这个整体因子积链的符号为 $(-1)^{\frac{a-(x+1)}{2}}$ 这个条件的首项基因子为 $1_{i_{x-1}}^{i_x} \cup 2_{i_{x-1}}^{i_x} \bullet \sqrt{\frac{1}{(l_{x-1}^{i_{x-1}})^2} - \frac{1}{4}}$ 的元素的 ${}^1x \to {}^10$ 型连环整体因子积链之和和所有满足因子积链的数积 a，$a \ge n+2$ 并与 $n+2$ 同奇偶，且这个整体因子积链的符号为 $(-1)^{\frac{a-(n+2)}{2}}$ 这个条件的首项基因子为 $1_{i_n} \cup 2_{i_n} \bullet \sqrt{\frac{1}{(l_n^{i_n})^2} - \frac{1}{4}}$ 的元素的 ${}^1(n+1) \to {}^10$ 型连环整体因子积链之和，${}_x^{x_i}B$ 的表达式和 F_n 纵坐标表达式所有满足因子积链的数积 a，$a \ge x+2$ 并与 $x+2$ 同奇偶，且这个整体因子积链的符号为 $(-1)^{\frac{a-(x+2)}{2}}$ 这个条件的首项基因子为 $1_{i_{x-1}}^{i_x} \cup 2_{i_{x-1}}^{i_x} \bullet \sqrt{\frac{1}{(l_{x-1}^{i_{x-1}})^2} - \frac{1}{4}}$ 的元素的 ${}^1x \to {}^10$ 型连环整体因子积链之和和所有满足因子积链的数积 a，$a \ge n+3$ 并与 $n+3$ 同奇偶，且这个整体因子积链符号为 $(-1)^{\frac{a-(n+3)}{2}}$ 这

个条件的首项基因子为$1_{i_n} \cup 2_{i_n} \bullet \sqrt{\frac{1}{(l_n^{i_n})^2}-\frac{1}{4}}$的元素的$^1(n+1) \to {}^1 0$型连环整体因子积链之和。

因此：当 x 为偶数时，坐标（${}_x^{x_i}A$，${}_x^{x_i}B$）的表达式可化为(当 n 为奇数时，F_n的表达式就是下面这个表达式中的 x 变为 n+1、$({}_x x_{2i_{x-1}-1}^{i_x} + {}_x x_{2i_{x-1}}^{i_x})$变为$(x_{2i_n-1} + x_{2i_n})$、$({}_x y_{2i_{x-1}-1}^{i_x} + {}_x y_{2i_{x-1}}^{i_x})$变为$(y_{2i_n-1} + y_{2i_n})$ 、 $({}_x x_{2i_{x-1}-1}^{i_x} - {}_x x_{2i_{x-1}}^{i_x})$ 变 为 $(x_{2i_n-1} - x_{2i_n})$ 、 $({}_x y_{2i_{x-1}-1}^{i_x} - {}_x y_{2i_{x-1}}^{i_x})$ 变 为 $(y_{2i_n-1} + y_{2i_n})$)：

$$
\left(\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{aligned}
&(({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))\,{}_1 x^{i_1}_{i_0}-\\
&3({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}+{}_2 y^{i_2}_{2i_1})\,{}_1 y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))\,{}_1 y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))\,{}_1 x^{i_1}_{i_0}+\cdots+\\
&(-3)({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))\,{}_1 y^{i_1}_{i_0}+\cdots+\\
&(-3)({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))\,{}_1 x^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}+{}_2 x^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))\,{}_1 y^{i_1}_{i_0})\cos\theta_{i_0}\\
&-\sqrt{3}(({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))\,{}_1 y^{i_1}_{i_0}+\\
&({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}+{}_2 y^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-2}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))\,{}_1 y^{i_1}_{i_0}+\cdots+\\
&({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))\,{}_1 x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-2}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))\,{}_1 x^{i_1}_{i_0})\sin\theta_{i_0}
\end{aligned}}{2\times 4^{x-1}}\right)\right)
$$

$$-\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet$$

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{array}{l}
((({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}- \\
3({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=2}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}+\cdots+ \\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}+\cdots+ \\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=2}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0})\sin\theta_{i_0} \\
+\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}+ \\
({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}+\cdots+ \\
({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0})\cos\theta_{i_0})
\end{array}}{4^{x-1}}\right)$$

$$-\cdots-$$

$$\sqrt{\frac{1}{{}^{i_{x-1}}_{x-1}A^2+{}^{i_{x-1}}_{x-1}B^2}-\frac{1}{4}}\bullet$$

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{aligned}&((({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}-\\
&3({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
&(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
&(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}+{}_{2}x^{i_2}_{2i_1})_1x^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0})\sin\theta_{i_0}\\
&+\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\\
&({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
&({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0})\cos\theta_{i_0})\end{aligned}}{4^{x-1}}\right)$$

$$+\cdots+(-1)^{\frac{x}{2}}\bullet$$

$$\left(\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}^{i_e}_{e}A^2+{}^{i_e}_{e}B^2}-\frac{1}{4}}\right)\bullet$$

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{array}{l}
((({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0}- \\
3({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}-{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0}+\cdots+ \\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0}+\cdots+ \\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}-{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1})_{1}x^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0})\sin\theta_{i_0} \\
+\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0}+ \\
({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0}+\cdots+ \\
({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0})\cos\theta_{i_0})
\end{array}}{2^x}\right)$$

,

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{array}{l}
(({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}-\\
3({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1y^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}+{}_{2}x^{i_2}_{2i_1})_1x^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0})\sin\theta_{i_0}\\
+\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\\
({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1x^{i_1}_{i_0}+\cdots+\\
(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0})\cos\theta_{i_0}
\end{array}}{2\times 4^{x-1}}\right)$$

$$
+\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet
$$

$$
\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{aligned}
&((({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}-\\
&3({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=2}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}+\cdots+\\
&(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}+\cdots+\\
&(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=2}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0})\cos\theta_{i_0}\\
&-\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}+\\
&({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}+\cdots+\\
&({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0})\sin\theta_{i_0})
\end{aligned}}{4^{x-1}}\right)
$$

$$+\cdots+$$

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\sqrt{\frac{1}{{}_{x-1}^{i_{x-1}}A^2+{}_{x-1}^{i_{x-1}}B^2}-\frac{1}{4}}\bullet
\left(\frac{\begin{gathered}
(((\,{}_{x}x_{2i_{x-1}-1}^{i_x}-{}_{x}x_{2i_{x-1}}^{i_x})(\prod_{i=1}^{x-2}({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}}))\,{}_{1}x_{i_0}^{i_1}-\\
3({}_{x}x_{2i_{x-1}-1}^{i_x}-{}_{x}x_{2i_{x-1}}^{i_x})(\prod_{i=2}^{x-2}({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}}))({}_{2}y_{2i_1-1}^{i_2}+{}_{2}y_{2i_1}^{i_2})\,{}_{1}y_{i_0}^{i_1}\\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x_{2i_{x-1}-1}^{i_x}-{}_{x}x_{2i_{x-1}}^{i_x})({}_{x-1}x_{2i_{x-2}-1}^{i_{x-1}}+{}_{x-1}x_{2i_{x-2}}^{i_{x-1}})(\prod_{i=1}^{x-3}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}}))\,{}_{1}y_{i_0}^{i_1}\\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x_{2i_{x-1}-1}^{i_x}-{}_{x}x_{2i_{x-1}}^{i_x})(\prod_{i=1}^{x-2}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}}))\,{}_{1}x_{i_0}^{i_1}+\cdots+\\
(-3)({}_{x}y_{2i_{x-1}-1}^{i_x}-{}_{x}y_{2i_{x-1}}^{i_x})(\prod_{i=1}^{x-2}({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}}))\,{}_{1}y_{i_0}^{i_1}+\cdots+\\
(-3)({}_{x}y_{2i_{x-1}-1}^{i_x}-{}_{x}y_{2i_{x-1}}^{i_x})({}_{x-1}y_{2i_{x-2}-1}^{i_{x-1}}+{}_{x-1}y_{2i_{x-2}}^{i_{x-1}})(\prod_{i=1}^{x-3}({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}}))\,{}_{1}x_{i_0}^{i_1}\\
+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}y_{2i_{x-1}-1}^{i_x}-{}_{x}y_{2i_{x-1}}^{i_x})(\prod_{i=2}^{x-2}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}}))({}_{2}x_{2i_1-1}^{i_2}+{}_{2}x_{2i_1}^{i_2})\,{}_{1}x_{i_0}^{i_1}\\
+\cdots+(-3)^{\frac{x}{2}}({}_{x}y_{2i_{x-1}-1}^{i_x}-{}_{x}y_{2i_{x-1}}^{i_x})(\prod_{i=1}^{x-2}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}}))\,{}_{1}y_{i_0}^{i_1})\cos\theta_{i_0}\\
-\sqrt{3}((\,{}_{x}x_{2i_{x-1}-1}^{i_x}-{}_{x}x_{2i_{x-1}}^{i_x})(\prod_{i=1}^{x-2}({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}}))\,{}_{1}y_{i_0}^{i_1}+\\
({}_{x}x_{2i_{x-1}-1}^{i_x}-{}_{x}x_{2i_{x-1}}^{i_x})(\prod_{i=2}^{x-2}({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}}))({}_{2}y_{2i_1-1}^{i_2}+{}_{2}y_{2i_1}^{i_2})\,{}_{1}x_{i_0}^{i_1}+\cdots+\\
(-3)^{\frac{x-2}{2}}({}_{x}x_{2i_{x-1}-1}^{i_x}-{}_{x}x_{2i_{x-1}}^{i_x})(\prod_{i=1}^{x-2}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}}))\,{}_{1}y_{i_0}^{i_1}+\cdots+\\
({}_{x}y_{2i_{x-1}-1}^{i_x}-{}_{x}y_{2i_{x-1}}^{i_x})(\prod_{i=1}^{x-2}({}_{i+1}x_{2i_i-1}^{i_{i+1}}+{}_{i+1}x_{2i_i}^{i_{i+1}}))\,{}_{1}x_{i_0}^{i_1}+\cdots+\\
(-3)^{\frac{x-2}{2}}({}_{x}y_{2i_{x-1}-1}^{i_x}-{}_{x}y_{2i_{x-1}}^{i_x})(\prod_{i=1}^{x-2}({}_{i+1}y_{2i_i-1}^{i_{i+1}}+{}_{i+1}y_{2i_i}^{i_{i+1}}))\,{}_{1}x_{i_0}^{i_1})\sin\theta_{i_0})
\end{gathered}}{4^{x-1}}\right)$$

$$
+\cdots+(-1)^{\frac{x-2}{2}}\bullet
$$

$$
\left(\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}_{e}^{i_e}A^2+{}_{e}^{i_e}B^2}-\frac{1}{4}}\right)\bullet
$$

$$
\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{aligned}
&((({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0}-\\
&3({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}-{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0}+\cdots+\\
&(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0}+\cdots+\\
&(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}-{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1})_{1}x^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0})\cos\theta_{i_0}\\
&-\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0}+\\
&({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-2}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}y^{i_1}_{i_0}+\cdots+\\
&({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}-{}_{i+1}x^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-2}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}-{}_{i+1}y^{i_{i+1}}_{2i_i}))_{1}x^{i_1}_{i_0})\sin\theta_{i_0})
\end{aligned}}{2^x}\right))
$$

当 x 为奇数时，坐标（${}^{x_i}_{x}A$, ${}^{x_i}_{x}B$）的表达式可化为(当 n 为偶数时，F_n 的表达式就是下面这个表达式中的 x 变为 n、$({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})$ 变为 $(x_{2i_n-1}+x_{2i_n})$、$({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})$ 变为 $(y_{2i_n-1}+y_{2i_n})$、$({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})$ 变为 $(x_{2i_n-1}-x_{2i_n})$、$({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})$ 变为 $(y_{2i_n-1}+y_{2i_n})$ ）：

$$
(\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}(\frac{\begin{array}{l}
(({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0}- \\
3({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}+{}_2 y^{i_2}_{2i_1})_1 y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-1}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0}+\cdots+ \\
(-3)({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0}+\cdots+ \\
(-3)({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0} \\
(-3)^{\frac{x-1}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0})\cos\theta_{i_0} \\
-\sqrt{3}(({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0}+ \\
({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}+{}_2 y^{i_2}_{2i_1})_1 x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-3}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-3}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0}+\cdots+ \\
({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-3}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}+{}_2 x^{i_2}_{2i_1})_1 x^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-1}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0})\sin\theta_{i_0}
\end{array}}{2\times 4^{x-1}})
$$

$$-\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet$$

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{array}{l}
((({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1})_{1}x^{i_1}_{i_0}- \\
3({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}y^{i_1}_{i_0}+\cdots+ \\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1})_{1}y^{i_1}_{i_0}+\cdots+ \\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=2}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1})_{1}x^{i_1}_{i_0} \\
(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}x^{i_1}_{i_0})\sin\theta_{i_0} \\
+\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1})_{1}y^{i_1}_{i_0}+ \\
({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=2}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}x^{i_1}_{i_0}+\cdots+ \\
({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1})_{1}x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-3}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}-{}_{2}x^{i_2}_{2i_1})_{1}x^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}-{}_{2}y^{i_2}_{2i_1})_{1}y^{i_1}_{i_0})\cos\theta_{i_0})
\end{array}}{4^{x-1}}\right)$$

$-\cdots-$

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\sqrt{\frac{1}{{}^{i_{x-1}}_{x-1}A^2+{}^{i_{x-1}}_{x-1}B^2}-\frac{1}{4}}\bullet\left(\frac{\begin{array}{l}
((({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}-\\
3({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1y^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}\\
(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0})\sin\theta_{i_0}\\
+\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\\
({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1x^{i_1}_{i_0}+\cdots+\\
(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
(-3)^{\frac{x-3}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}+{}_{2}x^{i_2}_{2i_1})_1x^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0})\cos\theta_{i_0})
\end{array}}{4^{x-1}}\right)$$

$$+\cdots+(-1)^{\frac{x-1}{2}}\bullet$$

$$\Big(\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}^{i_e}_{e}A^2+{}^{i_e}_{e}B^2}-\frac{1}{4}}\Big)\bullet$$

$$\sum_{i_n=1}^{n_n}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{gathered}
((({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}-\\
3({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1y^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}\\
(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0})\cos\theta_{i_0}\\
-\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\\
({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1x^{i_1}_{i_0}+\cdots+\\
(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
(-3)^{\frac{x-3}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}+{}_{2}x^{i_2}_{2i_1})_1x^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0})\sin\theta_{i_0})
\end{gathered}}{2^x}\right),$$

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\left(\frac{\begin{array}{l}
(({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i})){}_{1}x^{i_1}_{i_0}-\\
3({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1}){}_{1}y^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i})){}_{1}y^{i_1}_{i_0}+\cdots+\\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i})){}_{1}y^{i_1}_{i_0}+\cdots+\\
(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i})){}_{1}x^{i_1}_{i_0}\\
(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i})){}_{1}x^{i_1}_{i_0})\sin\theta_{i_0}\\
+\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i})){}_{1}y^{i_1}_{i_0}+\\
({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}+\cdots+\\
(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i})){}_{1}y^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}+{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i})){}_{1}x^{i_1}_{i_0}+\cdots+\\
({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i})){}_{1}x^{i_1}_{i_0}+\cdots+\\
(-3)^{\frac{x-3}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}+{}_{2}x^{i_2}_{2i_1}){}_{1}x^{i_1}_{i_0}\\
+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}+{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i})){}_{1}y^{i_1}_{i_0})\cos\theta_{i_0}
\end{array}}{2\times 4^{x-1}}\right)$$

$$+\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet\left(\frac{\begin{aligned}&((({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}-{}_2 x^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0}-\\&3({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}-{}_2 y^{i_2}_{2i_1})\,{}_1 y^{i_1}_{i_0}\\&+\cdots+(-3)^{\frac{x-1}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}-{}_2 y^{i_2}_{2i_1})\,{}_1 y^{i_1}_{i_0}+\cdots+\\&(-3)({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}-{}_2 x^{i_2}_{2i_1})\,{}_1 y^{i_1}_{i_0}+\cdots+\\&(-3)({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=2}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}-{}_2 x^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0}\\&(-3)^{\frac{x-1}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}-{}_2 y^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0})\cos\theta_{i_0}\\&-\sqrt{3}(({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}-{}_2 x^{i_2}_{2i_1})\,{}_1 y^{i_1}_{i_0}+\\&({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}-{}_2 y^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0}+\cdots+\\&(-3)^{\frac{x-3}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=2}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}-{}_2 y^{i_2}_{2i_1})\,{}_1 y^{i_1}_{i_0}\\&+\cdots+(-3)^{\frac{x-3}{2}}({}_x x^{i_x}_{2i_{x-1}-1}+{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}-{}_2 y^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0}+\cdots+\\&({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}-{}_2 x^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0}+\cdots+\\&(-3)^{\frac{x-3}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}-{}_2 x^{i_2}_{2i_1})\,{}_1 x^{i_1}_{i_0}\\&+\cdots+(-3)^{\frac{x-1}{2}}({}_x y^{i_x}_{2i_{x-1}-1}+{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}-{}_2 y^{i_2}_{2i_1})\,{}_1 y^{i_1}_{i_0})\sin\theta_{i_0})\end{aligned}}{4^{x-1}}\right)$$

$$
+\cdots+
$$

$$
\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}\sqrt{\frac{1}{{}^{i_{x-1}}_{x-1}A^2+{}^{i_{x-1}}_{x-1}B^2}-\frac{1}{4}}\bullet
\left(\frac{\begin{aligned}
&((({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}-\\
&3({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
&(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\cdots+\\
&(-3)({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}\\
&(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0})\cos\theta_{i_0}\\
&-\sqrt{3}(({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}+\\
&({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_{2}y^{i_2}_{2i_1-1}+{}_{2}y^{i_2}_{2i_1})_1x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-3}{2}}({}_{x}x^{i_x}_{2i_{x-1}-1}-{}_{x}x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
&({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1x^{i_1}_{i_0}+\cdots+\\
&(-3)^{\frac{x-3}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_{2}x^{i_2}_{2i_1-1}+{}_{2}x^{i_2}_{2i_1})_1x^{x-1}_{i_0}\\
&+\cdots+(-3)^{\frac{x-1}{2}}({}_{x}y^{i_x}_{2i_{x-1}-1}-{}_{x}y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1y^{i_1}_{i_0})\sin\theta_{i_0})
\end{aligned}}{4^{x-1}}\right)
$$

$$+\cdots+(-1)^{\frac{x-1}{2}}\bullet$$

$$(\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}_e^{i_e}A^2+{}_e^{i_e}B^2}-\frac{1}{4}})\bullet$$

$$\sum_{i_x=1}^{n_x}\cdots\sum_{i_0=1}^{n_0}(\frac{\begin{gathered}((({}_x x^{i_x}_{2i_{x-1}-1}-{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0}- \\
3({}_x x^{i_x}_{2i_{x-1}-1}-{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}+{}_2 y^{i_2}_{2i_1})_1 y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-1}{2}}({}_x x^{i_x}_{2i_{x-1}-1}-{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0}+\cdots+ \\
(-3)({}_x y^{i_x}_{2i_{x-1}-1}-{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0}+\cdots+ \\
(-3)({}_x y^{i_x}_{2i_{x-1}-1}-{}_x y^{i_x}_{2i_{x-1}})({}_{x-1}y^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}y^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0} \\
(-3)^{\frac{x-1}{2}}({}_x y^{i_x}_{2i_{x-1}-1}-{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0})\sin\theta_{i_0} \\
+\sqrt{3}(({}_x x^{i_x}_{2i_{x-1}-1}-{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0}+ \\
({}_x x^{i_x}_{2i_{x-1}-1}-{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))({}_2 y^{i_2}_{2i_1-1}+{}_2 y^{i_2}_{2i_1})_1 x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-3}{2}}({}_x x^{i_x}_{2i_{x-1}-1}-{}_x x^{i_x}_{2i_{x-1}})({}_{x-1}x^{i_{x-1}}_{2i_{x-2}-1}+{}_{x-1}x^{i_{x-1}}_{2i_{x-2}})(\prod_{i=1}^{x-3}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-3}{2}}({}_x x^{i_x}_{2i_{x-1}-1}-{}_x x^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0}+\cdots+ \\
({}_x y^{i_x}_{2i_{x-1}-1}-{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}x^{i_{i+1}}_{2i_i-1}+{}_{i+1}x^{i_{i+1}}_{2i_i}))_1 x^{i_1}_{i_0}+\cdots+ \\
(-3)^{\frac{x-3}{2}}({}_x y^{i_x}_{2i_{x-1}-1}-{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=2}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))({}_2 x^{i_2}_{2i_1-1}+{}_2 x^{i_2}_{2i_1})_1 x^{i_1}_{i_0} \\
+\cdots+(-3)^{\frac{x-1}{2}}({}_x y^{i_x}_{2i_{x-1}-1}-{}_x y^{i_x}_{2i_{x-1}})(\prod_{i=1}^{x-2}({}_{i+1}y^{i_{i+1}}_{2i_i-1}+{}_{i+1}y^{i_{i+1}}_{2i_i}))_1 y^{i_1}_{i_0})\cos\theta_{i_0})\end{gathered}}{2^x}))$$

第三、前人工作的总结：

从两点出发作图时生锈圆规的能力和普通规尺是等价的的原因是第一次一级，二级，三级作图的二级作图中，利用生锈圆规从坐标差为

$$\begin{array}{l}(\frac{x}{2}(x_1\cos\theta_1+\cdots+x_n\cos\theta_n)-\frac{\sqrt{3}y}{2}(x_1\sin\theta_1+\cdots+x_n\sin\theta_n),\\ \frac{x}{2}(x_1\sin\theta_1+\cdots+x_n\sin\theta_n)+\frac{\sqrt{3}y}{2}(x_1\cos\theta_1+\cdots+x_n\cos\theta_n))\end{array}$$

型的两点和坐标差为

$$\begin{array}{l}(\frac{x}{2}(x_1\cos\theta_1+\cdots+x_n\cos\theta_n)+\frac{\sqrt{3}y}{2}(x_1\sin\theta_1+\cdots+x_n\sin\theta_n),\\ \frac{x}{2}(x_1\sin\theta_1+\cdots+x_n\sin\theta_n)-\frac{\sqrt{3}y}{2}(x_1\cos\theta_1+\cdots+x_n\cos\theta_n))\end{array}$$

型的两点出发进行作图。

（$x,y,x_1,\cdots,x_n\in Z$，且 x 与 y 同奇偶）

令 $x_1\cos\theta_1+\cdots+x_n\cos\theta_n=A,x_1\sin\theta_1+\cdots+x_n\sin\theta_n=B$,可知第一次一级，二级，三级作图的三级作图中，可得到

$$\begin{array}{l}(\frac{xA}{4}-\frac{\sqrt{3}yB}{4}-\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet(\frac{xB}{2}+\frac{\sqrt{3}yA}{2}),\\ \frac{xB}{4}+\frac{\sqrt{3}yA}{4}+\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet(\frac{xA}{2}-\frac{\sqrt{3}yB}{2}))\end{array}$$

和

$$\begin{array}{l}(\frac{xA}{4}+\frac{\sqrt{3}yB}{4}-\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet(\frac{xB}{2}-\frac{\sqrt{3}yA}{2}),\\ \frac{xB}{4}-\frac{\sqrt{3}yA}{4}+\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet(\frac{xA}{2}+\frac{\sqrt{3}yB}{2}))\end{array}$$

这两个（A1,B1)，可推得第二次一级，二级，三级作图的二级作图中，可得坐标差为：

$$\begin{array}{l}(\frac{xA}{2}-\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet xB,\\ \frac{xB}{2}+\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet xA)\end{array}$$

型的两点，可推得第二次一级，二级，三级作图的三级作图中，可得这些（A1,B1) 为

$$(\frac{cxA}{4}-\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet\frac{cxB}{2}-\sqrt{\frac{x^2+3y^2}{4c^2x^2}-\frac{1}{4}}\bullet(\frac{cxB}{2}+\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet cxA),$$
$$\frac{cxB}{4}+\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet\frac{cxA}{2}+\sqrt{\frac{x^2+3y^2}{4c^2x^2}-\frac{1}{4}}\bullet(\frac{cxA}{2}-\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet cxB)\quad ,$$

$$(\frac{cxA}{4}-\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet\frac{cxB}{2}+\sqrt{\frac{x^2+3y^2}{4c^2x^2}-\frac{1}{4}}\bullet(\frac{cxB}{2}+\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet cxA),$$
$$\frac{cxB}{4}+\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet\frac{cxA}{2}-\sqrt{\frac{x^2+3y^2}{4c^2x^2}-\frac{1}{4}}\bullet(\frac{cxA}{2}-\sqrt{\frac{4}{(x^2+3y^2)(A^2+B^2)}-\frac{1}{4}}\bullet cxB)\quad ,$$

$\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet$（注：$c\in Z,c\neq 0$，且取值满足所需条件。）

即第 3 级$\sqrt{\frac{1}{l^2}-\frac{1}{4}}$型因子为可以一个$\sqrt{\frac{x^2+3y^2}{4c^2x^2}-\frac{1}{4}}$这样的实数。

总之，主要是反复利用生锈圆规从坐标差为$(xA\mp yB,xB\pm yA)$型的两点和坐标差为$(xA\pm yB,xB\mp yA)$型的两点出发作图，得到这些（A1,B1）：

$$(\frac{xA}{2}\mp\sqrt{\frac{4}{(x^2+y^2)(A^2+B^2)}-\frac{1}{4}}\bullet cxB-\sqrt{\frac{x^2+y^2}{4c^2x^2}-\frac{1}{4}}\bullet(cxB\pm\sqrt{\frac{4}{(x^2+y^2)(A^2+B^2)}-\frac{1}{4}}\bullet 2cxA),$$
$$\frac{cxB}{2}\pm\sqrt{\frac{4}{(x^2+y^2)(A^2+B^2)}-\frac{1}{4}}\bullet cxA+\sqrt{\frac{x^2+y^2}{4c^2x^2}-\frac{1}{4}}\bullet(cxA\mp\sqrt{\frac{4}{(x^2+y^2)(A^2+B^2)}-\frac{1}{4}}\bullet 2cxB)\quad ,$$

$$(\frac{xA}{2}\mp\sqrt{\frac{4}{(x^2+y^2)(A^2+B^2)}-\frac{1}{4}}\bullet cxB+\sqrt{\frac{x^2+y^2}{4c^2x^2}-\frac{1}{4}}\bullet(cxB\pm\sqrt{\frac{4}{(x^2+y^2)(A^2+B^2)}-\frac{1}{4}}\bullet 2cxA),$$
$$\frac{cxB}{2}\pm\sqrt{\frac{4}{(x^2+y^2)(A^2+B^2)}-\frac{1}{4}}\bullet cxA-\sqrt{\frac{x^2+y^2}{4c^2x^2}-\frac{1}{4}}\bullet(cxA\mp\sqrt{\frac{4}{(x^2+y^2)(A^2+B^2)}-\frac{1}{4}}\bullet 2cxB)$$

$\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet$（注：$c\in Z,c\neq 0$，且取值满足所需条件。）

后，进行二级作图。通过这样的方法得到利用生锈圆规从两点出发作图所求作的点。最后可得到这样的结论，利用普通规尺从两点出发作图所可以作出的点，利用生锈圆规也可以

做出。

第四、生锈圆规作图方程定理及其推导：

通过上面的内容可知：当 x 为偶数时，${}^{x_i}_{x}A$ 为：

$$\sum_{i=1}^{n_0}\begin{array}{l}(\frac{{}_1x_i}{2\times4^{x-1}}\cos\theta_i-\frac{\sqrt{3}\,{}_1y_i}{2\times4^{x-1}}\sin\theta_i-\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_i}{4^{x-1}}\sin\theta_i+\frac{\sqrt{3}\,{}_2y_i}{4^{x-1}}\cos\theta_i)\\+\cdots\cdots+(-1)^{\frac{x}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}^{i_e}_{e}A^2+{}^{i_e}_{e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_i}{2^x}\sin\theta_i+\frac{\sqrt{3}\,{}_{2^{x-1}}y_i}{2^x}\cos\theta_i)\end{array}$$

，${}^{x_i}_{x}B$ 为：

$$\sum_{i=1}^{n_0}\begin{array}{l}(\frac{{}_1x_i}{2\times4^{x-1}}\sin\theta_i+\frac{\sqrt{3}\,{}_1y_i}{2\times4^{x-1}}\cos\theta_i+\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_i}{4^{x-1}}\cos\theta_i-\frac{\sqrt{3}\,{}_2y_i}{4^{x-1}}\sin\theta_i)\\+\cdots\cdots+(-1)^{\frac{x-2}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}^{i_e}_{e}A^2+{}^{i_e}_{e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_i}{2^x}\cos\theta_i-\frac{\sqrt{3}\,{}_{2^{x-1}}y_i}{2^x}\sin\theta_i)\end{array}$$

$({}_1x_i,{}_1y_i,{}_2x_i,{}_2y_i,\cdots,{}_{2^{x-1}}x_i,{}_{2^{x-1}}y_i\in Z)$。

当 x 为奇数时，${}^{x_i}_{x}A$ 为：

$$\sum_{i=1}^{n_0}\begin{array}{l}(\frac{{}_1x_i}{2\times4^{x-1}}\cos\theta_i-\frac{\sqrt{3}\,{}_1y_i}{2\times4^{x-1}}\sin\theta_i-\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_i}{4^{x-1}}\sin\theta_i+\frac{\sqrt{3}\,{}_2y_i}{4^{x-1}}\cos\theta_i)\\+\cdots\cdots+(-1)^{\frac{x-1}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}^{i_e}_{e}A^2+{}^{i_e}_{e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_i}{2^x}\cos\theta_i-\frac{\sqrt{3}\,{}_{2^{x-1}}y_i}{2^x}\sin\theta_i)\end{array}$$

，${}^{x_i}_{x}B$ 为：

$$\sum_{i=1}^{n_0}\begin{array}{l}(\frac{{}_1x_i}{2\times4^{x-1}}\sin\theta_i+\frac{\sqrt{3}\,{}_1y_i}{2\times4^{x-1}}\cos\theta_i+\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_i}{4^{x-1}}\cos\theta_i-\frac{\sqrt{3}\,{}_2y_i}{4^{x-1}}\sin\theta_i)\\+\cdots\cdots+(-1)^{\frac{x-1}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}^{i_e}_{e}A^2+{}^{i_e}_{e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_i}{2^x}\sin\theta_i+\frac{\sqrt{3}\,{}_{2^{x-1}}y_i}{2^x}\cos\theta_i)\end{array}$$

$({}_1x_i,{}_1y_i,{}_2x_i,{}_2y_i,\cdots,{}_{2^{x-1}}x_i,{}_{2^{x-1}}y_i\in Z)$。

由此可得：当 x 为偶数时，${}^{x_i}_{x}A^2+{}^{x_i}_{x}B^2$ 为

$$\left(\sum_{i=1}^{n_0}\begin{matrix}(\frac{{}_1x_i}{2\times 4^{x-1}}\cos\theta_i-\frac{\sqrt{3}\,{}_1y_i}{2\times 4^{x-1}}\sin\theta_i-\sqrt{\frac{1}{{}_1^{i_1}A^2+{}_1^{i_1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_i}{4^{x-1}}\sin\theta_i+\frac{\sqrt{3}\,{}_2y_i}{4^{x-1}}\cos\theta_i)\\ +\cdots\cdots+(-1)^{\frac{x}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}_e^{i_e}A^2+{}_e^{i_e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_i}{2^x}\sin\theta_i+\frac{\sqrt{3}\,{}_{2^{x-1}}y_i}{2^x}\cos\theta_i)\end{matrix}\right)^2+$$

$$\left(\sum_{i=1}^{n_0}\begin{matrix}(\frac{{}_1x_i}{2\times 4^{x-1}}\sin\theta_i+\frac{\sqrt{3}\,{}_1y_i}{2\times 4^{x-1}}\cos\theta_i+\sqrt{\frac{1}{{}_1^{i_1}A^2+{}_1^{i_1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_i}{4^{x-1}}\cos\theta_i-\frac{\sqrt{3}\,{}_2y_i}{4^{x-1}}\sin\theta_i)\\ +\cdots\cdots+(-1)^{\frac{x-2}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}_e^{i_e}A^2+{}_e^{i_e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_i}{2^x}\cos\theta_i-\frac{\sqrt{3}\,{}_{2^{x-1}}y_i}{2^x}\sin\theta_i)\end{matrix}\right)^2$$

即

$$\sum_{1\le i,j\le n_0}\begin{matrix}(\frac{{}_1x_c}{4^{2x-1}}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}\,{}_1y_c}{4^{2x-1}}\sin(\theta_i-\theta_j)+\sqrt{\frac{1}{{}_1^{i_1}A^2+{}_1^{i_1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_c}{4^{2x-2}}\sin(\theta_i-\theta_j)+\frac{\sqrt{3}\,{}_2y_c}{4^{2x-2}}\cos(\theta_i-\theta_j))\\ +\cdots\cdots+(-1)^{\frac{x}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}_e^{i_e}A^2+{}_e^{i_e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_c}{2^{3x-1}}\sin(\theta_i-\theta_j)+\frac{\sqrt{3}\,{}_{2^{x-1}}y_c}{2^{3x-1}}\cos(\theta_i-\theta_j))+\cdots\cdots+\\ (-1)^x\bullet((\prod_{1\le e,f\le x-1}\sqrt{\frac{1}{{}_e^{i_e}A^2+{}_e^{i_e}B^2}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}_f^{i_f}A^2+{}_f^{i_f}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^x}x_c}{4^{2x}}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}\,{}_{2^x}y_c}{4^{2x}}\sin(\theta_i-\theta_j))\end{matrix}$$

$({}_1x_c, {}_1y_c, {}_2x_c, {}_2y_c, \cdots, {}_{2^x}x_c, {}_{2^x}y_c \in Z)$

当 x 为奇数时， ${}_x^{x_i}A^2+{}_x^{x_i}B^2$ 为

$$\left(\sum_{i=1}^{n_0}\begin{matrix}(\frac{{}_1x_i}{2\times 4^{x-1}}\cos\theta_i-\frac{\sqrt{3}\,{}_1y_i}{2\times 4^{x-1}}\sin\theta_i-\sqrt{\frac{1}{{}_1^{i_1}A^2+{}_1^{i_1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_i}{4^{x-1}}\sin\theta_i+\frac{\sqrt{3}\,{}_2y_i}{4^{x-1}}\cos\theta_i)\\ +\cdots\cdots+(-1)^{\frac{x-1}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}_e^{i_e}A^2+{}_e^{i_e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_i}{2^x}\cos\theta_i-\frac{\sqrt{3}\,{}_{2^{x-1}}y_i}{2^x}\sin\theta_i)\end{matrix}\right)^2+$$

$$\left(\sum_{i=1}^{n_0}\begin{matrix}(\frac{{}_1x_i}{2\times 4^{x-1}}\sin\theta_i+\frac{\sqrt{3}\,{}_1y_i}{2\times 4^{x-1}}\cos\theta_i+\sqrt{\frac{1}{{}_1^{i_1}A^2+{}_1^{i_1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_i}{4^{x-1}}\cos\theta_i-\frac{\sqrt{3}\,{}_2y_i}{4^{x-1}}\sin\theta_i)\\ +\cdots\cdots+(-1)^{\frac{x-1}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}_e^{i_e}A^2+{}_e^{i_e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_i}{2^x}\sin\theta_i+\frac{\sqrt{3}\,{}_{2^{x-1}}y_i}{2^x}\cos\theta_i)\end{matrix}\right)^2$$

即

$$\sum_{1\le i,j\le n_0}\begin{aligned}&(\frac{{}_1x_c}{4^{2x-1}}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}{}_1y_c}{4^{2x-1}}\sin(\theta_i-\theta_j)+\sqrt{\frac{1}{{}^{i_1}_{1}A^2+{}^{i_1}_{1}B^2}-\frac{1}{4}}\bullet(\frac{{}_2x_c}{4^{2x-2}}\sin(\theta_i-\theta_j)+\frac{\sqrt{3}{}_2y_c}{4^{2x-2}}\cos(\theta_i-\theta_j))\\&+\cdots\cdots+(-1)^{\frac{x}{2}}\bullet((\prod_{e=1}^{x-1}\sqrt{\frac{1}{{}^{i_e}_{e}A^2+{}^{i_e}_{e}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^{x-1}}x_c}{2^{3x-1}}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}{}_{2^{x-1}}y_c}{2^{3x-1}}\sin(\theta_i-\theta_j))+\cdots\cdots+\\&(-1)^x\bullet((\prod_{1\le e,f\le x-1}\sqrt{\frac{1}{{}^{i_e}_{e}A^2+{}^{i_e}_{e}B^2}-\frac{1}{4}}\bullet\sqrt{\frac{1}{{}^{i_f}_{f}A^2+{}^{i_f}_{f}B^2}-\frac{1}{4}})\bullet(\frac{{}_{2^x}x_c}{4^x}\sin(\theta_i-\theta_j)+\frac{\sqrt{3}{}_{2^x}y_c}{4^x}\cos(\theta_i-\theta_j))\end{aligned}$$

$({}_1x_c,{}_1y_c,{}_2x_c,{}_2y_c,\cdots,{}_{2^x}x_c,{}_{2^x}y_c\in Z)$ 。注：若 ${}_dx_c=({}_ax_i\bullet{}_bx_j+3{}_ay_i\bullet{}_by_j)$ ，则 ${}_dy_c=({}_ax_i\bullet{}_by_j-{}_ay_i\bullet{}_bx_j)$ $(d,a,b\in N^+,1\le d\le 2^x,1\le a\le 2^{x-1},1\le b\le 2^{x-1})$ 。

又因为：当 x=1 时，${}^{x_i}_{x}A^2+{}^{x_i}_{x}B^2$ 为 $\sum_{1\le i,j\le n_0}(\frac{{}_1x_c}{2}\cos(\theta_i-\theta_j)+\frac{\sqrt{3}{}_1y_c}{2}\sin(\theta_i-\theta_j))$，当 i=j 时，$\frac{{}_1x_c}{2}\cos(\theta_i-\theta_j)=\frac{{}_1x_c}{2},\frac{\sqrt{3}{}_1y_c}{2}\sin(\theta_i-\theta_j)=0$。

$$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^2+{}^{i_{i_e}}_{i_e}B^2}-\frac{1}{4}}=$$

$$\sqrt{\begin{aligned}&\frac{1}{({}^{i_{i_1}}_{i_1}A^2+{}^{i_{i_1}}_{i_1}B^2)\bullet\cdots\bullet({}^{i_{i_x}}_{i_x}A^2+{}^{i_{i_x}}_{i_x}B^2)}+\cdots+\\&(-\frac{1}{4})^y\bullet(\frac{1}{({}^{i_{i_1}}_{i_1}A^2+{}^{i_{i_1}}_{i_1}B^2)\bullet\cdots\bullet({}^{i_{i_{x-y}}}_{i_{x-y}}A^2+{}^{i_{i_{x-y}}}_{i_{x-y}}B^2)}+\cdots+\frac{1}{({}^{i_{i_{y+1}}}_{i_{y+1}}A^2+{}^{i_{i_{y+1}}}_{i_{y+1}}B^2)\bullet\cdots\bullet({}^{i_{i_x}}_{i_x}A^2+{}^{i_{i_x}}_{i_x}B^2)})\\&+\cdots+(-\frac{1}{4})^x\end{aligned}}$$

$(i_e\in N^+,\le i_e\le n)$

由此可推得：生锈圆规作图方程定理：F_n 的表达式是一个关于常数与 $\cos\theta_i,\sin\theta_i$ 进行加，减，乘，除，平方和开平方运算搅在一起的解析几何方程，F_n 的表达式中的代数式

$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$是对常数，$\cos(\theta_i-\theta_j)$，$\sin(\theta_i-\theta_j)$进行加，减，乘，除，平方和开平方的混合运算代数式，由于加，减，乘，除，平方和开平方运算的规定，可推得：一、代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$代数式中出现$\sin(\theta_i-\theta_j)$的系数不为 **0**，则代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$中必出现常数$\sqrt{3}$，即$F_n$的表达式必出现常数$\sqrt{3}$。代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$分子中的$\cos(\theta_i-\theta_j)$，$\sin(\theta_i-\theta_j)$和分母中的$\cos(\theta_i-\theta_j)$，$\sin(\theta_i-\theta_j)$所对应的系数和所处于结构位置一样（当 **i=j** 时，除外），与代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$相乘的代数式中的中的$\cos\theta_i,\sin\theta_i$在代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$中一定存在。二、与代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$相乘的代数式中的$\cos\theta_i$，$\sin\theta_i$在代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$中除去常数后一定存在。代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$中除去常数后存在的$\cos\theta_i$，$\sin\theta_i$在与代数式$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^{2}+{}^{i_{i_e}}_{i_e}B^{2}}-\frac{1}{4}}$相乘的代数式中不一定存在。（注$\cos(\theta_i-\theta_j)=\cos\theta_i\cos\theta_j+\sin\theta_i\sin\theta_j$，$\sin(\theta_i-\theta_j)=\sin\theta_i\cos\theta_j-\cos\theta_i\sin\theta_j$）

由于当 i=j 时：$\cos(\theta_i - \theta_j) = 1$，$\sin(\theta_i - \theta_j) = 0$，所以，$\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^2 + {}^{i_{i_e}}_{i_e}B^2} - \frac{1}{4}}$ 可以是一个实数。这是前人为什么可以得到“利用普通规尺从两点出发作图所可以作出的点，利用生锈圆规也可以做出”这个结论的原因。

五、应用解决新问题：

问题（1）：已知 A,B,C 三点，能否作出点 D,使得 AC=AD,BC=BD。

无法作出点 D。证明：

设点 A 坐标为 (0,0)，点 B 坐标为 $(\sum_{i=1}^{2n_1}\cos\theta_i, \sum_{i=1}^{2n_1}\sin\theta_i)$，点 C 的坐标为 $(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i, \sum_{i=2n_1+1}^{2n_2}\sin\theta_i)$。

则所求点 D 点的坐标为：

$$\left(\frac{-(\sum_{i=1}^{2n_1}\sin\theta_i)^2\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i) - 2(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i) + (\sum_{i=1}^{2n_1}\cos\theta_i)^2\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)}{(\sum_{i=1}^{2n_1}\cos\theta_i)^2 + (\sum_{i=1}^{2n_1}\sin\theta_i)^2},\right.$$

$$\left.\frac{-(\sum_{i=1}^{2n_1}\sin\theta_i)^2\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i) - 2(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i) - (\sum_{i=1}^{2n_1}\cos\theta_i)^2\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)}{(\sum_{i=1}^{2n_1}\cos\theta_i)^2 + (\sum_{i=1}^{2n_1}\sin\theta_i)^2}\right)。$$

此点坐标是无法得到的。分子中的 $\cos(\theta_i - \theta_j)$，$\sin(\theta_i - \theta_j)$ 和分母中的 $\cos(\theta_i - \theta_j)$，

$\sin(\theta_i-\theta_j)$所对应的系数和所处于结构位置不一样，因此违反了生锈圆规作图方程定理。故此点不能做出。

问题（2）：已知 A，B，C，D 四点（四个点中任意三点不共线），能否作出直线 AB 与 CD 的交点 E？AC 与 BD 的交点 F？AD 与 BC 交点 G？

无法作出点 E,点 F,点 G。证明：

设点 A 坐标为(0,0)，点 B 坐标为$(\sum_{i=1}^{2n_1}\cos\theta_i,\sum_{i=1}^{2n_1}\sin\theta_i)$，点 C 的坐标为$(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i,\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)$，点 D 的坐标为$(\sum_{i=2n_2+1}^{2n_3}\cos\theta_i,\sum_{i=2n_2+1}^{2n_3}\sin\theta_i)$，所需要的辅助点所确定的（A1， B1）为$(\cos\theta_{2n_3+1},\sin\theta_{2n_3+1})\cdots(\cos\theta_{2n},\sin\theta_{2n})$。则 AC 与 BD 的交点 F 的坐标为

$$\left(\frac{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)\bullet(\sum_{\substack{1\le i\le 2n_1,\\ 2n_2+1\le j\le 2n_3}}\sin(\theta_i-\theta_j))}{\sum_{\substack{2n_1+1\le i\le 2n_2,\\ 2n_2+1\le j\le 2n_3}}\sin(\theta_i-\theta_j)},\frac{(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)\bullet(\sum_{\substack{1\le i\le 2n_1,\\ 2n_2+1\le j\le 2n_3}}\sin(\theta_i-\theta_j))}{\sum_{\substack{2n_1+1\le i\le 2n_2,\\ 2n_2+1\le j\le 2n_3}}\sin(\theta_i-\theta_j)}\right)$$。

此点坐标是无法得到的。分子中的$\cos(\theta_i-\theta_j)$， $\sin(\theta_i-\theta_j)$和分母中的$\cos(\theta_i-\theta_j)$，$\sin(\theta_i-\theta_j)$所对应的系数和所处于结构位置不一样，违反了生锈圆规作图方程定理。故此点不能做出

问题（3）：仅利用$\triangle ABC$的三个顶点 A,B,C，能否作出$\triangle ABC$的外心点 D、内心点 E、垂心点 F、三个垂足点G_1,点G_2,点G_3、三个旁心点H_1,点H_2,点H_3。

无法作出点 D、点 E、点 F、点G_1,点G_2,点G_3、点H_1,点H_2,点H_3。证明：

设点 A 坐标为(0,0)，点 B 坐标为$(\sum_{i=1}^{2n_1}\cos\theta_i,\sum_{i=1}^{2n_1}\sin\theta_i)$，点 C 的坐标为$(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i,\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)$。则所求点 D 的坐标为：

$$\left(\frac{(\sum_{i=1}^{2n_1}\sin\theta_i)\bullet((\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2)-(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)\bullet((\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2)}{2(\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)-2(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)},\right.$$

$$\left.\frac{-(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet((\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2)+(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)\bullet((\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2)}{2(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)-2(\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)}\right)$$

、所求点 E 的坐标为：

$$\left(\frac{\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)+\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}\bullet(\sum_{i=1}^{2n_1}\cos\theta_i)}{\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}+\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}+\sqrt{((\sum_{i=1}^{2n_1}\cos\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))^2+((\sum_{i=1}^{2n_1}\sin\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i))^2}},\right.$$

$$\left.\frac{\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)+\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}\bullet(\sum_{i=1}^{2n_1}\sin\theta_i)}{\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}+\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}+\sqrt{((\sum_{i=1}^{2n_1}\cos\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))^2+((\sum_{i=1}^{2n_1}\sin\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i))^2}}\right)$$

、所求点 F 的坐标为：

$$\left(\frac{((\sum_{i=1}^{2n_1}\sin\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i))\bullet((\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)+(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))}{((\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)-(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i))},\right.$$

$$\left.\frac{-((\sum_{i=1}^{2n_1}\cos\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))\bullet((\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)+(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))}{((\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)-(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i))}\right)$$。

所求点 G_1 的坐标为：

$$\left(\frac{(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet((\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)+(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))}{((\sum_{i=1}^{2n_1}\sin\theta_i)^2+(\sum_{i=1}^{2n_1}\cos\theta_i)^2)},\right.$$

$$\left.\frac{(\sum_{i=1}^{2n_1}\sin\theta_i)\bullet((\sum_{i=1}^{2n_1}\sin\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)+(\sum_{i=1}^{2n_1}\cos\theta_i)\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))}{((\sum_{i=1}^{2n_1}\sin\theta_i)^2+(\sum_{i=1}^{2n_1}\cos\theta_i)^2)}\right)$$、

所求点 H_1 的坐标为：

$$\left(\frac{\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)+\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}\bullet(\sum_{i=1}^{2n_1}\cos\theta_i)}{\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}+\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}-\sqrt{((\sum_{i=1}^{2n_1}\cos\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))^2+((\sum_{i=1}^{2n_1}\sin\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i))^2}},\right.$$

$$\frac{\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}\bullet(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)+\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}\bullet(\sum_{i=1}^{2n_1}\sin\theta_i)}{\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}+\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}-\sqrt{((\sum_{i=1}^{2n_1}\cos\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i))^2+((\sum_{i=1}^{2n_1}\sin\theta_i)-(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i))^2}})。$$

点 D 的坐标，点 F 的坐标，点 G_1 的坐标无法做出的原因是：**分子中的 $\cos(\theta_i-\theta_j)$，$\sin(\theta_i-\theta_j)$ 和分母中的 $\cos(\theta_i-\theta_j)$，$\sin(\theta_i-\theta_j)$ 所对应的系数和所处于结构位置不一样，违反了生锈圆规作图方程定理。**故点 D，点 F，点 G_1，点 G_2，点 G_3 不能做出。

点 E 的坐标、点 H_1 的坐标无法做出的原因是：**由生锈圆规作图方程定理：与代数式 $\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^2+{}^{i_{i_e}}_{i_e}B^2}-\frac{1}{4}}$ 相乘的代数式中的 $\cos\theta_i$，$\sin\theta_i$ 在代数式 $\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^2+{}^{i_{i_e}}_{i_e}B^2}-\frac{1}{4}}$ 中除去常数后一定存在。**代数式 $\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^2+{}^{i_{i_e}}_{i_e}B^2}-\frac{1}{4}}$ 中除去常数后存在的 $\cos\theta_i$，$\sin\theta_i$ 在与代数式 $\prod_{e=1}^{x}\sqrt{\frac{1}{{}^{i_{i_e}}_{i_e}A^2+{}^{i_{i_e}}_{i_e}B^2}-\frac{1}{4}}$ 相乘的代数式中不一定存在。可知：分子中 $\sqrt{(\sum_{i=1}^{2n_1}\cos\theta_i)^2+(\sum_{i=1}^{2n_1}\sin\theta_i)^2}\bullet(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)$ 和 $\sqrt{(\sum_{i=2n_1+1}^{2n_2}\cos\theta_i)^2+(\sum_{i=2n_1+1}^{2n_2}\sin\theta_i)^2}\bullet(\sum_{i=1}^{2n_1}\cos\theta_i)$ 是不可能得到的，所以故点 E，点 H_1，点 H_2，点 H_3 不能做出。

六、作图定理：

作图定理:若平面上存在 n+1 个点,分别设为:$(a,b),(a+x_1,b+y_1),\cdots,(a+x_n,b+y_n)$，且不对$(x_1,y_1),\cdots,(x_n,y_n)$进行特殊限定，则对于坐标为：$(a+\sum_j^m(A_j\bullet(\sum_{i=1}^n {}_ja_i\bullet x_i+{}_jb_i\bullet y_i),\ b+\sum_j^m(B_j\bullet(\sum_{i=1}^n {}_jc_i\bullet x_i+{}_jd_i\bullet y_i))$ (${}_ja_i$，${}_jb_i$，${}_jc_i$，${}_jd_i\in R,A_j$，B_j 均是关于${}_je_i\bullet x_i$，${}_jf_i\bullet y_i$进行有限次运算，乘方，开方运算得到的代数式，且不能化为一个常数。（${}_je_i$，${}_jf_i\in R$)) 这样的点利用生锈圆规作图均不能作出，利用尺规作图或圆规作图部分点可以做出。

证明：因为代数式$\prod_{e=1}^x\sqrt{\frac{1}{{}_{i_e}^{i_{i_e}}A^2+{}_{i_e}^{i_{i_e}}B^2}-\frac{1}{4}}$中一定含有常数，所以生锈圆规作图方程$F_n$当$n>1$时，除去$F_1$后的代数式均是含有常数，所以对于坐标为：$(a+\sum_j^m(A_j\bullet(\sum_{i=1}^n {}_ja_i\bullet x_i+{}_jb_i\bullet y_i),\ b+\sum_j^m(B_j\bullet(\sum_{i=1}^n {}_jc_i\bullet x_i+{}_jd_i\bullet y_i))$ (${}_ja_i$，${}_jb_i$，${}_jc_i$，${}_jd_i\in R,A_j$，B_j 均是关于${}_je_i\bullet x_i$，${}_jf_i\bullet y_i$进行有限次运算，乘方，开方运算得到的代数式，且不能化为一个常数。（${}_je_i$，${}_jf_i\in R$)) 这样的点利用生锈圆规作图均不能作出的。尺规作图和圆规作图均可作出：已知 A,B,C 三点，作出点 D,使得 AC=AD,BC=BD。已知 A，B，C，D 四点（四个点中任意三点不共线），作出直线 AB 与 CD 的交点 E、直线 AC 与 BD 的交点 F、直线 AD 与 BC 交点 G。仅利用$\triangle ABC$的三个顶点 A,B,C，作出$\triangle ABC$的外心点 D、内心点 E、垂心点 F、三个垂足点G_1,点G_2,点G_3、三个旁心点H_1,点H_2,点H_3。所以，一定可以作出坐标为：$(a+\sum_j^m(A_j\bullet(\sum_{i=1}^n {}_ja_i\bullet x_i+{}_jb_i\bullet y_i),\ b+\sum_j^m(B_j\bullet(\sum_{i=1}^n {}_jc_i\bullet x_i+{}_jd_i\bullet y_i))$ (${}_ja_i$，${}_jb_i$，${}_jc_i$，${}_jd_i\in R,A_j$，B_j 均是关于${}_je_i\bullet x_i$，${}_jf_i\bullet y_i$进行有限次运算，乘方，开方运算得到的代数式，且不能化为一个常数。（${}_je_i$，${}_jf_i\in R$)) 这样的点的其中一部分。

七、关于 3n+1 猜想的规定及 3n+1 方程的构造：

规定：1、任意一个自然数 x，用 $\sum_{i=0}^{n} x_i 2^i (n \in N^+)$ 且 x$\geq 2^n$，对于任意 x_i 的取值均为 0 或 1 的形式表示。下文，对于关于形如 $\sum_{i=0}^{n} x_i 2^i (n \in N^+)$ 的表达式中，对于任意 x_i 的取值均为 0 或 1。

2、设一个奇数 a 乘 3 以再加 1 后得到偶数 3a+1，偶数 3a+1 除以 $2^n (n \in N^+)$ 得到新的奇数 b 的这个过程为 1 次程序运算。

3、若一个奇数 a 是经过 $n(n \in N)$ 次程序运算得到奇数 1，则称 $a \bullet 2^m (m \in N)$ 均是经过 n 次程序运算得到奇数 1 的数。则称 $(\frac{a \bullet 2^m - 1}{3}) \bullet 2^n (n \in N)$ 均是经过 $n+1$ 次程序运算得到奇数 1 的数。

证明：原猜想可划归为：任何一个正整数 a，经过有限次程序运算总可以得到奇数 1。

经过 0 次程序运算可以得到奇数 1 的正整数：$2^{n_1} (n \in N^+)$，由 $1 \bullet 2^{n_1} (n_1 \in N)$ 化简而得。

经过 1 次程序运算可以得到奇数 1 的正整数：$(\sum_{i=0}^{n_1} 2^{2i}) \bullet 2^{n_2} (n_1, n_2 \in N)$，由 $(\frac{2^{2(n_1+1)} - 1}{3}) \bullet 2^{n_2} (n_1, n_2 \in N)$ 化简而得。

经过 2 次程序运算可以得到奇数 1 的正整数：① $(\sum_{i=0}^{n_2} 2^{2i} + \sum_{i=0}^{n_1} 2^{2(3i+n_2+2)} + 2^{2(3i+n_2+2)+1} + 2^{2(3i+n_2+2)+2}) \bullet 2^{n_3} (n_1, n_2, n_3 \in N)$，由 $(\frac{(\frac{2^{2(3n_1+2)} - 1}{3}) \bullet 2^{2(n_2+1)} - 1}{3}) \bullet 2^{n_3} (n_1, n_2, n_3 \in N)$ 化简而得。②

$((\frac{2^{[n_2-1]}}{2^{n_2-1}}) \bullet \sum_{i=0}^{n_2-1} 2^{2i} + 2^{2n_2} + 2^{2n_2+1} + \sum_{i=0}^{n_1} 2^{2(3i+n_2+2)+1} + 2^{2(3i+n_2+2)+2} + 2^{2(3i+n_2+2)+3}) \bullet 2^{n_3} (n_1, n_2, n_3 \in N)$，由 $(\frac{(\frac{2^{2(3n_1+3)} - 1}{3}) \bullet 2^{2n_2+1} - 1}{3}) \bullet 2^{n_3} (n_1, n_2, n_3 \in N)$ 化简而得。

……………………………

通过以上内容，可推得规律：经过 n 次程序运算可以得到奇数 1 的正整数的公式是一个方程为：

$$((\prod_{i=2}^{n}\frac{2^{[m_i-1+x_i]}}{2^{m_i-1+x_i}})\bullet\sum_{i=0}^{m_n-1+x_n}2^{2i}+(\frac{2^{2(m_n+x_n)}\bullet\left|x_n\right|+(\sum_{i=m_n+1}^{m_n+m_{n-1}-1+x_{n-1}}2^{2i+x_n})\bullet\frac{2^{[m_{n-1}-2+x_{n-1}]}}{2^{m_{n-1}-2+x_{n-1}}}}{3})\bullet(\prod_{i=2}^{n-1}\frac{2^{[m_i-1+x_i]}}{2^{m_i-1+x_i}})+$$

$$(\frac{2^{2(m_n+m_{n-1}+x_{n-1})+x_n}\bullet\left|x_{n-1}\right|+(\sum_{i=m_{n-1}+1}^{m_{n-1}+m_{n-2}-1+x_{n-2}}2^{2i+x_n+x_{n-1}})\bullet\frac{2^{[m_{n-2}-2+x_{n-2}]}}{2^{m_{n-2}-2+x_{n-2}}}-3\bullet(1-\frac{2^{[m_{n-1}-1+x_{n-1}]}}{2^{m_{n-1}-1+x_{n-1}}})}{9})\bullet(\prod_{i=2}^{n-2}\frac{2^{[m_i-1+x_i]}}{2^{m_i-1+x_i}})+$$

$$(\sum_{y=3}^{n-2}(\frac{2^{(\sum_{j=n-y+1}^{n}(2m_j+x_j))+x_{n-y+1}}\bullet\left|x_{n-y+1}\right|+(\sum_{i=(\sum_{j=n-y+1}^{n}m_i)+1}^{(\sum_{j=n-y+1}^{n}m_i)+m_{n-y}-1+x_{n-y}}2^{2i+(\sum_{j=n-y+1}^{n}x_i)})\bullet\frac{2^{[m_{n-y}-2+x_{n-y}]}}{2^{m_{n-y}-2+x_{n-y}}}-(\sum_{i=1}^{y-2}3^i\cdot 2^{\sum_{j=n+i+1-y}^{n}(2m_j+x_j)}+3^{y-1})\bullet(1-\frac{2^{[m_{n-y+1}-1+x_{n-y+1}]}}{2^{m_{n-y+1}-1+x_{n-y+1}}})}{3^y})\bullet(\prod_{i=2}^{n-y}\frac{2^{[m_i-1+x_i]}}{2^{m_i-1+x_i}})+$$

$$(\frac{2^{(\sum_{i=2}^{n}(2m_i+x_i))+x_2}\bullet\left|x_2\right|+\sum_{j=\sum_{i=2}^{n}(m_i)+1}^{\sum_{i=2}^{n}(m_i)+m_1}2^{2j+\sum_{i=2}^{n}(x_i)}-(\sum_{i=1}^{n-3}3^i\cdot 2^{\sum_{j=i+2}^{n}(2m_j+x_j)}+3^{n-2})\bullet(1-\frac{2^{[m_2-1+x_2]}}{2^{m_2-1+x_2}})}{3^{n-1}}))\bullet 2^{m_{n+1}}$$

（$m_1>1, m_1\in N^+, m_j\in N^+, x_i\in\{0,-1\}\,(j\in\{2,\cdots,n+1\}, i\in\{2,\cdots,n\}$）中的所有正整数。即规律的推导公式可用数学归纳法证明：

1、由于经过 1 次程序运算可以得到奇数 1 的正整数为 $(\sum_{i=0}^{n_1}2^{2i})\bullet 2^{n_1}\,(n_1,n_2\in N)$，满足上式，故当 k=1 时原命题成立。2、假设 k 为 n 时原命题成立。3、则当 k 为 n+1 时：①当 $m_b=1, x_b=-1, b\in\{2,\cdots,n\}$ 且设任意一个小于 b 的正整数 c，则

$m_c+x_c>0$时：特设$x_1=1$，则公式为：

$$(\frac{(\sum_{y=n-b+1}^{n-1}(\frac{2^{(\sum_{j=n-y+1}^{n}(2m_j+x_j))+x_{n-y+1}}\bullet|x_{n-y+1}|+(\sum_{i=(\sum_{j=n-y+1}^{n}m_i)+1}^{(\sum_{j=n-y+1}^{n}m_i)+m_{n-y}-1+x_{n-y}}2^{2i+(\sum_{j=n-y+1}^{n}x_i)})\bullet\frac{2^{[m_{n-y}-2+x_{n-y}]}}{2^{m_{n-y}-2+x_{n-y}}}-((\sum_{i=1}^{n-b-1}3^i\cdot 2^{\sum_{j=b+i}^{n}(2m_j+x_j)})\bullet\frac{3^{[n-2-b]}}{3^{n-2-b}}+3^{n-b})\bullet(\frac{3^{[n-1-b]}}{3^{n-1-b}})}{3^{n-b+1}}))\bullet 2^{2m_{n+1}+x_{n+1}}-1}{3})\bullet 2^{m_{n+2}}$$

。②当任意一个正整数$i,i\in\{2,\cdots,n\}$，则$m_i+x_i>0$时：则公式为（已化简）：

$$(\frac{\sum_{i=0}^{m_n-1+x_n}2^{2i}}{3}+\frac{2^{2(m_n+x_n)}\bullet|x_n|+(\sum_{i=m_n+1}^{m_n+m_{n-1}-1+x_{n-1}}2^{2i+x_n})\bullet\frac{2^{[m_{n-1}-2+x_{n-1}]}}{2^{m_{n-1}-2+x_{n-1}}}}{9}+$$

$$\sum_{y=2}^{n-2}(\frac{2^{(\sum_{j=n-y+1}^{n}(2m_j+x_j))+x_{n-y+1}}\bullet|x_{n-y+1}|+(\sum_{i=(\sum_{j=n-y+1}^{n}m_i)+1}^{(\sum_{j=n-y+1}^{n}m_i)+m_{n-y}-1+x_{n-y}}2^{2i+(\sum_{j=n-y+1}^{n}x_i)})\bullet\frac{2^{[m_{n-y}-2+x_{n-y}]}}{2^{m_{n-y}-2+x_{n-y}}}}{3^{y+1}})+$$

$$(\frac{2^{(\sum_{i=2}^{n}(2m_i+x_i))+x_2}\bullet|x_2|+\sum_{j=\sum_{i=2}^{n}(m_i)+1}^{\sum_{i=2}^{n}(m_i)+m_1}2^{2j+\sum_{i=2}^{n}(x_i)}}{3^n})\bullet 2^{2m_{n+1}+x_{n+1}}-1)\bullet 2^{m_{n+2}}$$

（当$m_{n+1}=1,x_{n+1}=-1$时，即$2m_{n+1}+x_{n+1}=1$时，由于$\frac{2^0\bullet 2-1}{3}=\frac{2^0}{3}$，$m_n-1+x_n$可能等于0，所以

$$\frac{(\sum_{i=0}^{m_n-1+x_n} 2^{2i})\bullet 2-1}{3}=\frac{2^0+(\sum_{i=1+1}^{1+m_n-1+x_n} 2^{2i-1})\bullet\frac{2^{[m_n-2+x_n]}}{2^{m_n-2+x_n}}}{3}=\frac{2^{2(m_{n+1}+x_{n+1})}\bullet|x_{n+1}|+(\sum_{i=m_{n+1}+1}^{m_{n+1}+m_n-1+x_n} 2^{2i+x_n})\bullet\frac{2^{[m_n-2+x_n]}}{2^{m_n-2+x_n}}}{3}$$

。当 $m_{n+1}=1, x_{n+1}=0$，或 $m_{n+1}=2, x_{n+1}=-1$ 时，即 $2m_{n+1}+x_{n+1}=2$ 或 $2m_{n+1}+x_{n+1}=3$ 时，由于 $\frac{2^0\bullet 2^2-1}{3}=2^0$，$\frac{2^0\bullet 2^3-1}{3}=2^0+\frac{2^2}{3}$，所以

$$\frac{(\sum_{i=0}^{m_n-1+x_n} 2^{2i})\bullet 2^2-1}{3}=2^0+\frac{(\sum_{i=1+1}^{1+m_n-1+x_n} 2^{2i})\bullet\frac{2^{[m_n-2+x_n]}}{2^{m_n-2+x_n}}}{3}=$$

$$\sum_{i=0}^{m_{n+1}-1+x_{n+1}} 2^{2i}+\frac{2^{2(m_{n+1}+x_{n+1})}\bullet|x_{n+1}|+(\sum_{i=m_{n+1}+1}^{m_{n+1}+m_n-1+x_n} 2^{2i+x_n})\bullet\frac{2^{[m_n-2+x_n]}}{2^{m_n-2+x_n}}}{3},$$

$$\frac{(\sum_{i=0}^{m_n-1+x_n} 2^{2i})\bullet 2^3-1}{3}=2^0+\frac{2^2+(\sum_{i=2+1}^{2+m_n-1+x_n} 2^{2i-1})\bullet\frac{2^{[m_n-2+x_n]}}{2^{m_n-2+x_n}}}{3}=$$

$$\sum_{i=0}^{m_{n+1}-1+x_{n+1}} 2^{2i}+\frac{2^{2(m_{n+1}+x_{n+1})}\bullet|x_{n+1}|+(\sum_{i=m_{n+1}+1}^{m_{n+1}+m_n-1+x_n} 2^{2i+x_n})\bullet\frac{2^{[m_n-2+x_n]}}{2^{m_n-2+x_n}}}{3}$$

。将这两式和并为一个公式为：

$$((\prod_{i=2}^{n+1}\frac{2^{[m_i-1+x_i]}}{2^{m_i-1+x_i}})\bullet\sum_{i=0}^{m_{n+1}-1+x_{n+1}} 2^{2i}+(\frac{2^{2(m_{n+1}+x_{n+1})}\bullet|x_{n+1}|+(\sum_{i=m_{n+1}+1}^{m_{n+1}+m_n-1+x_n} 2^{2i+x_{n+1}})\bullet\frac{2^{[m_n-2+x_n]}}{2^{m_n-2+x_n}}}{3})\bullet(\prod_{i=2}^{n}\frac{2^{[m_i-1+x_i]}}{2^{m_i-1+x_i}})+$$

$$(\frac{2^{2(m_{n+1}+m_n+x_n)+x_{n+1}}\bullet|x_n|+(\sum_{i=m_n+1}^{m_n+m_{n-1}-1+x_{n-1}} 2^{2i+x_{n+1}+x_n})\bullet\frac{2^{[m_{n-1}-2+x_{n-1}]}}{2^{m_{n-1}-2+x_{n-1}}}-3\bullet(1-\frac{2^{[m_n-1+x_n]}}{2^{m_n-1+x_n}})}{9})\bullet(\prod_{i=2}^{n-1}\frac{2^{[m_i-1+x_i]}}{2^{m_i-1+x_i}})+$$

$$2^{(\sum_{j=n+1-y+1}^{n+1}(2m_j+x_j))+x_{n+1-y+1}}\bullet\left|x_{n+1-y+1}\right|+(\sum_{i=(\sum_{j=n+1-y+1}^{n+1}m_i)+1}^{(\sum_{j=n+1-y+1}^{n+1}m_i)+m_{n+1-y}-1+x_{n+1-y}}2^{2i+(\sum_{j=n+1-y+1}^{n+1}x_i)})\bullet\frac{2^{\left[m_{n+1-y}-2+x_{n+1-y}\right]}}{2^{m_{n+1-y}-2+x_{n+1-y}}}$$

$$(\sum_{y=3}^{n-1}(\frac{-(\sum_{i=1}^{y-2}3^i\cdot 2^{\sum_{j=n+1+i+1-y}^{n+1}(2m_j+x_j)}+3^{y-1})\bullet(1-\frac{2^{\left[m_{n+1-y+1}-1+x_{n+1-y+1}\right]}}{2^{m_{n+1-y+1}-1+x_{n+1-y+1}}})}{3^y})\bullet(\prod_{i=2}^{n+1-y}\frac{2^{\left[m_i-1+x_i\right]}}{2^{m_i-1+x_i}})+$$

$$(\frac{2^{(\sum_{i=2}^{n+1}(2m_i+x_i))+x_2}\bullet\left|x_2\right|+\sum_{j=\sum_{i=2}^{n+1}(m_i)+1}^{\sum_{i=2}^{n+1}(m_i)+m_1}2^{2j+\sum_{i=2}^{n+1}(x_i)}-(\sum_{i=1}^{n+1-3}3^i\cdot 2^{\sum_{j=i+2}^{n+1}(2m_j+x_j)}+3^{n+1-2})\bullet(1-\frac{2^{\left[m_2-1+x_2\right]}}{2^{m_2-1+x_2}})}{3^n})\bullet 2^{m_{n+2}}$$

（$m_1>1, m_1\in N^+, m_j\in N^+, x_i\in\{0,-1\}$ ($j\in\{2,\cdots,n+2\}, i\in\{2,\cdots,n+1\}$)）中的所有正整数。即 k 为 n+1 时原命题成立。故原命题成立。

八、3n+1 方程：

要 证 明 3n+1 猜 想 只 需 解 决 方 程 ：

$$((\prod_{i=2}^{p}\frac{2^{\left[m_i-1+x_i\right]}}{2^{m_i-1+x_i}})\bullet\sum_{i=0}^{m_p-1+x_p}2^{2i}+(\frac{2^{2(m_p+x_p)}\bullet\left|x_p\right|+(\sum_{i=m_p+1}^{m_p+m_{p-1}-1+x_{p-1}}2^{2i+x_p})\bullet\frac{2^{\left[m_{p-1}-2+x_{p-1}\right]}}{2^{m_{p-1}-2+x_{p-1}}}}{3})\bullet(\prod_{i=2}^{p-1}\frac{2^{\left[m_i-1+x_i\right]}}{2^{m_i-1+x_i}})+$$

$$(\frac{2^{2(m_p+m_{p-1}+x_{p-1})+x_p}\bullet\left|x_{p-1}\right|+(\sum_{i=m_{p-1}+1}^{m_{p-1}+m_{p-2}-1+x_{p-2}}2^{2i+x_p+x_{p-1}})\bullet\frac{2^{\left[m_{p-2}-2+x_{p-2}\right]}}{2^{m_{p-2}-2+x_{p-2}}}-3\bullet(1-\frac{2^{\left[m_{p-1}-1+x_{p-1}\right]}}{2^{m_{p-1}-1+x_{p-1}}})}{9})\bullet(\prod_{i=2}^{p-2}\frac{2^{\left[m_i-1+x_i\right]}}{2^{m_i-1+x_i}})+$$

$$\Big(\sum_{y=3}^{p-2}\Big(\frac{2^{(\sum_{j=p-y+1}^{p}(2m_j+x_j))+x_{p-y+1}}\bullet\left|x_{p-y+1}\right|+\Big(\sum_{i=(\sum_{j=p-y+1}^{p}m_i)+1}^{(\sum_{j=p-y+1}^{p}m_i)+m_{p-y}-1+x_{p-y}}2^{2i+(\sum_{j=p-y+1}^{p}x_i)}\Big)\bullet\frac{2^{\left[m_{p-y}-2+x_{p-y}\right]}}{2^{m_{p-y}-2+x_{p-y}}}-\Big(\sum_{i=1}^{y-2}3^i\cdot 2^{\sum_{j=p+i+1-y}^{p}(2m_j+x_j)}+3^{y-1}\Big)\bullet\Big(1-\frac{2^{\left[m_{p-y+1}-1+x_{p-y+1}\right]}}{2^{m_{p-y+1}-1+x_{p-y+1}}}\Big)}{3^y}\Big)\bullet\Big(\prod_{i=2}^{p-y}\frac{2^{\left[m_i-1+x_i\right]}}{2^{m_i-1+x_i}}\Big)+$$

$$\Big(\frac{2^{(\sum_{i=2}^{p}(2m_i+x_i))+x_2}\bullet\left|x_2\right|+\sum_{j=\sum_{i=2}^{p}(m_i)+1}^{\sum_{i=2}^{p}(m_i)+m_1}2^{2j+\sum_{i=2}^{p}(x_i)}-\Big(\sum_{i=1}^{p-3}3^i\cdot 2^{\sum_{j=i+2}^{p}(2m_j+x_j)}+3^{p-2}\Big)\bullet\Big(1-\frac{2^{\left[m_2-1+x_2\right]}}{2^{m_2-1+x_2}}\Big)}{3^{p-1}}\Big)\Big)-$$

$$\Big(\Big(\prod_{i=2}^{q}\frac{2^{\left[n_i-1+w_i\right]}}{2^{n_i-1+w_i}}\Big)\bullet\sum_{i=0}^{n_q-1+w_q}2^{2i}+\Big(\frac{2^{2(n_q+w_q)}\bullet\left|w_q\right|+\Big(\sum_{i=n_q+1}^{n_q+n_{q-1}-1+w_{q-1}}2^{2i+w_q}\Big)\bullet\frac{2^{\left[n_{q-1}-2+w_{q-1}\right]}}{2^{n_{q-1}-2+w_{q-1}}}}{3}\Big)\bullet\Big(\prod_{i=2}^{q-1}\frac{2^{\left[n_i-1+w_i\right]}}{2^{n_i-1+w_i}}\Big)+$$

$$\Big(\frac{2^{2(n_q+n_{q-1}+w_{q-1})+w_q}\bullet\left|w_{q-1}\right|+\Big(\sum_{i=n_{q-1}+1}^{n_{q-1}+n_{q-2}-1+w_{q-2}}2^{2i+w_q+w_{q-1}}\Big)\bullet\frac{2^{\left[n_{q-2}-2+w_{q-2}\right]}}{2^{n_{q-2}-2+w_{q-2}}}-3\bullet\Big(1-\frac{2^{\left[n_{q-1}-1+w_{q-1}\right]}}{2^{n_{q-1}-1+w_{q-1}}}\Big)}{9}\Big)\bullet\Big(\prod_{i=2}^{q-2}\frac{2^{\left[n_i-1+w_i\right]}}{2^{n_i-1+w_i}}\Big)+$$

$$\Big(\sum_{y=3}^{q-2}\Big(\frac{2^{(\sum_{j=q-y+1}^{q}(2n_j+w_j))+w_{q-y+1}}\bullet\left|w_{q-y+1}\right|+\Big(\sum_{i=(\sum_{j=q-y+1}^{q}n_i)+1}^{(\sum_{j=q-y+1}^{q}n_i)+n_{q-y}-1+w_{q-y}}2^{2i+(\sum_{j=q-y+1}^{q}w_i)}\Big)\bullet\frac{2^{\left[n_{q-y}-2+w_{q-y}\right]}}{2^{n_{q-y}-2+w_{q-y}}}-\Big(\sum_{i=1}^{y-2}3^i\cdot 2^{\sum_{j=q+i+1-y}^{q}(2n_j+w_j)}+3^{y-1}\Big)\bullet\Big(1-\frac{2^{\left[n_{q-y+1}-1+w_{q-y+1}\right]}}{2^{n_{q-y+1}-1+w_{q-y+1}}}\Big)}{3^y}\Big)\bullet\Big(\prod_{i=2}^{q-y}\frac{2^{\left[n_i-1+w_i\right]}}{2^{n_i-1+w_i}}\Big)+$$

$$\Big(\frac{2^{(\sum_{i=2}^{q}(2n_i+w_i))+w_2}\bullet\left|x_2\right|+\sum_{j=\sum_{i=2}^{q}(n_i)+1}^{\sum_{i=2}^{q}(n_i)+n_1}2^{2j+\sum_{i=2}^{q}(w_i)}-\Big(\sum_{i=1}^{q-3}3^i\cdot 2^{\sum_{j=i+2}^{q}(2n_j+w_j)}+3^{q-2}\Big)\bullet\Big(1-\frac{2^{\left[n_2-1+w_2\right]}}{2^{n_2-1+w_2}}\Big)}{3^{q-1}}\Big)\Big)=2$$

（m_1、$n_1 > 1, m_1$、$n_1 \in N^+, m_j$、$n_k \in N^+, x_i$、$w_h \in \{0, -1\}$（j、$i \in \{2, \cdots, p\}, k$、$h \in \{2, \cdots, q\}$））

此为 3n+1 方程。

九、3n+1 猜想的问题展望：

目前作者认为对于 3n+1 方程的解决尚需时日，一步解决 3n+1 方程是很困难的。因为 p 和 q 的取值可以取任意自然数，所以 3n+1 方程的解决将会是方程学的一个很大的进展。可以先进行一些验算，从中找出规律来解决 3n+1 方程。3n+1 方程的解决，即 3n+1 猜想的解决可能会对计算机科学的发展有极大的促进作用，且一旦解决就可以得到问题：任给一个正整数 n，如果 n 能被 a 整除，就将它变为 n/a，如果除后不能再整除，则将它乘 b 加 c（即 bn+c）。不断重复这样的运算，经过有限步后，一定可以得到 d 吗？的答案。3n+1 猜想是一个很值得研究的数学问题，3n+1 方程是一个很有研究价值的方程。

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